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AN ELEMENTARY TREATISE ON THE GEOMETRY OF CONICS.

Alexander Ziwet

AN ELEMENTARY TREATISE

ON THE

GEOMETRY OF CONICS.

BY

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PREFACE.

THIS work contains elementary proofs of the principal properties of Conics, and is intended for students who proceed to the study of the subject after finishing the first six books of Euclid; the curves have not, therefore, been defined as the sections of a cone, although that method has the sanction of history and antiquity in its favour; and for the same reason, no use has been made of the method of projections.

As regards the arrangement of the subject, I have thought it best to devote separate chapters to the parabola, the ellipse, and the hyperbola. The plan of starting with a chapter on general conics, in which some fundamental propositions are proved by methods applicable to all the three curves, has no doubt the advantage of securing an appearance of brevity. But, I believe, beginners find the subject more intelligible when the properties of the three curves are discussed separately. Besides, in the other method students, and even writers of text-books, are apt to overlook the necessity of modifying an argument on account of the fundamental difference in the figures of the several curves; see, for instance, Chap. II., Prop. x., and Chap. III., Prop. ix., which are ordinarily proved by identically the same argument. Also, as the properties of the hyperbola are proved, wherever possible, by the same methods as the corresponding properties of the ellipse, it is obvious that this arrangement does not tend to increase the work of the student.

As to the propositions included in each chapter and their sequence, I have not been able to adopt wholly the scheme of any previous writer; but I venture to hope that the book includes all the classical propositions on the subject, arranged in their proper logical order. Every attempt has been made to render the proofs simple and easily intelligible, though I have never sacrificed accuracy to brevity. Thus, for instance, I have not followed the practice of referring to a proposition when the truth of its converse is really assumed—a practice which has, in at least one instance, led to a remarkable error in the treatment of conjugate diameters in a famous text-book. Nor have I attempted to secure a fictitious appearance of conciseness by adding to each proposition a list of corollaries by no means less important than the proposition itself, and freely using them for the purpose of deducing subsequent propositions.

The exercises, of which there are about eight hundred, have been selected with great care; more than six hundred of these are placed under the different propositions from which they may be deduced; they are for the most part of an elementary character, and have been carefully graduated. Hints and solutions have been liberally added, and these, it is hoped, will prove materially helpful to the student, and render the subject attractive. The attention of the student has also been directed to various methods of graphically describing the curves, including those used in practice by draughtsmen, and some very neat problems have been added from Newton, Book I., Sections iv. and v.

At the end of the table of contents will be found a course of reading suitable for beginners.

CALCUTTA,
19*th April*, 1893.

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Propositions marked with an asterisk may be omitted by the beginner. This would leave for a first course of reading—

Chap. I.—Props. i.-ii., iv.-vii., x.-xii., xiv., xvii.-xix., xxiii.-xxv., ... (16)

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GEOMETRY OF CONICS.

GEOMETRY OF CONICS.

INTRODUCTION.

A CONIC is a curve traced by a point which moves in a plane containing a fixed point and a fixed straight line, in such a way that its distance from the fixed point is in a constant ratio to its perpendicular distance from the fixed straight line.

The fixed point is called the FOCUS.

The fixed straight line is called the DIRECTRIX.

The constant ratio is called the ECCENTRICITY, and is usually represented by the letter e.

When the eccentricity is equal to unity, the Conic is called a PARABOLA ($e=1$).

When the eccentricity is less than unity, the Conic is called an ELLIPSE ($e<1$).

When the eccentricity is greater than unity, the Conic is called a HYPERBOLA ($e>1$).

The straight line drawn through the focus perpendicular to the directrix is called the AXIS of the Conic.

The point (or points) in which the axis intersects the Conic is called the VERTEX.

The Conics are so called from the circumstance that they are, and were originally studied as, the plane sections of the surface of a right circular cone, which is a surface formed by the revolution of a right-angled triangle about one of its sides. This conception does not lead to the simplest way of investigating the properties of Conics, as it necessitates a knowledge of the geometry of solids. In order to restrict the discussion of these curves to the domain of plane geometry, they have been defined as above.

The Conics are said to have been discovered by Menaechmus, a Greek mathematician who flourished about B.C. 350, and were accordingly called after him the "*Menaechmian Triads.*" They were first systematically studied by Apollonius of Perga (B.C. 247-205).

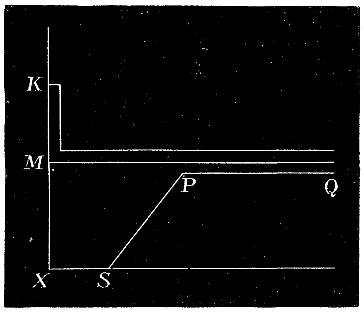
CHAPTER I.

THE PARABOLA.

DESCRIPTION OF THE CURVE.

We have seen that the eccentricity of the parabola is unity, that is, the distance of any point on it from the focus is equal to its perpendicular distance from the directrix.

The parabola may be mechanically constructed in the following manner.



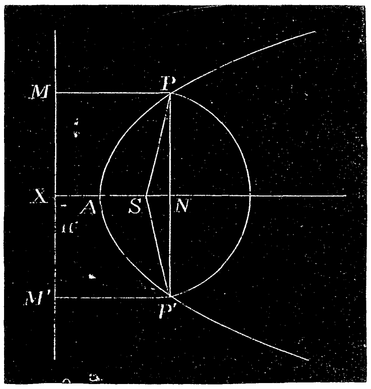
Let S be the focus and MX the directrix; and let a rigid bar KMQ, of which the portions KM and MQ are at right angles to each other, having a string of the same length as MQ, fastened at the end Q, be made to slide parallel to the axis SX with the end M on the directrix; then if the other end of the string be fastened at the focus S, and the string be kept stretched by means of the point of a pencil at P, in contact with the bar, it is evident that the point P will trace out a parabola, since SP is always equal to PM.

Ex. A point moves so that the sum of its distances from a fixed point and a fixed straight line is constant. Show that it describes a parabola.

In the above figure, the sum of the distances of P from S and the straight line through Q parallel to XK is evidently constant.

Proposition I.

Given the focus and the directrix of a parabola, to determine any number of points on it.



Let S be the focus and MXM' the directrix. Through S draw SX perpendicular to the directrix, and bisect SX in A; then A is a point on the parabola, since $SA = AX$.

Take any point N in SX or SX produced. Through N draw PNP' perpendicular to XN; with centre S and radius equal to XN, describe a circle cutting PNP' at P and P'; then P and P' shall be points on the parabola.

Draw PM and $P'M'$ perpendicular to the directrix.

Then $PS = XN$, by construction, and $PM = XN$, being opposite sides of a rectangle; therefore $PS = PM$. Similarly it may be shown that $P'S = P'M'$. Therefore P and P' are points on the parabola.

In like manner, by taking any other point in SX, any number of points on the curve may be determined.

Ex. 1. The parabola is symmetrical with respect to its axis.

This follows from the fact that PP' is bisected at right angles by XS.

Def. A curve is said to be *symmetrical with respect to a straight line*, if, corresponding to any point on the curve, there is another point on the curve on the other side of the straight line, such that the chord joining them is bisected at right angles by the straight line.

Ex. 2. *Alternative Construction*—Join the focus S to any point M on the directrix; draw MP at right angles to the directrix, and make the angle MSP equal to the angle SMP. P is a point on the parabola.

Ex. 3. *Alternative Construction.*—Bisect SM in E, and draw EP perpendicular to SM, meeting MP in P. P is a point on the parabola.

For another construction, see Prop. X., Ex. 3.

Ex. 4. Describe a parabola of which the focus and vertex are given.

Ex. 5. Given the focus S, and two points P, Q on the parabola, construct it.

The directrix will be a common tangent to the two circles described, with centre S and radii SP, SQ respectively.

Ex. 6. The distance of any point inside the parabola from the focus is less than its distance from the directrix.

Ex. 7. The distance of any point outside the parabola from the focus is greater than its distance from the directrix.

Ex. 8. A straight line parallel to the axis of a parabola meets the curve in one point only.

Ex. 9. There is no limit to the distance to which the parabola may extend on both sides of the axis, so that the parabola is not a *closed* curve.

It is obvious that the point N may be taken *anywhere* on the axis.

Ex. 10. Any two right lines drawn from the focus to the curve on opposite sides of the axis, and equally inclined to it, are equal; and conversely.

Ex. 11. If SM meets in Y the straight line drawn through A perpendicular to the axis, $SY = YM$, and PY is at right angles to SM and bisects the angle SPM.

Ex. 12. If SZ is drawn at right angles to SP to meet the directrix in Z, PZ bisects the angle SPM.

Ex. 13. PSp is a right line passing through the focus and meeting the parabola in P and p. PM and pm are perpendicular to the directrix. Show that MSm is a right angle.

Ex. 14. The locus of the centre of a circle which passes through a given point and touches a given straight line is a parabola.

Ex. 15. The locus of the centre of a circle which touches a given circle and a given straight line is a parabola.

The focus is the centre of the given circle, and the directrix a right line parallel to the given one at a distance from it equal to the radius of the given circle.

Ex. 16. PSp is a straight line through the focus S, cutting the parabola in P and p. PN, pn are drawn at right angles to the axis. Prove that $AN \cdot An = AS^2$.

Ex. 17. Given the directrix and two points on the curve, construct it. Show that, in general, two parabolas satisfy the conditions.

Ex. 18. If from a point P of a circle, PC be drawn to the centre C, and R be the middle point of the chord PQ drawn parallel to a fixed diameter ACB; then the locus of the intersection of CP and AR is a parabola.

The focus will be at C, and the directrix will be the tangent to the circle at A.

PROPERTIES OF CHORDS.

Def. The *chord* (QQ') of a conic is the finite straight line joining any two points (Q, Q') on the curve.

Def. A *focal chord* (PSp) is any chord drawn through the focus (S).

Def. The *latus rectum* (LL') of a conic is the focal chord drawn at right angles to the axis.

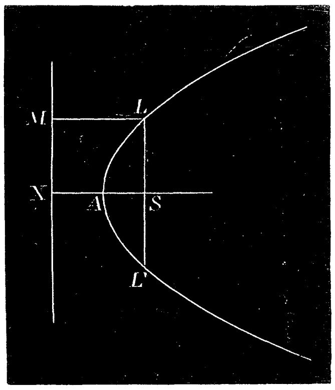
Def. The *focal distance* (SP) of a point (P) on a conic is its distance from the focus.

Def. The *ordinate* (PN) of a point (P) on a conic is the perpendicular from the point on the axis.

Def. The *abscissa* (AN) of a point (P) on a parabola, with respect to the axis, is the portion of the axis between the vertex and the ordinate of the point.

PROPOSITION II.

The latus rectum of a parabola is equal to four times the distance of the focus from the vertex ($LL' = 4AS$).



Let LSL' be the latus rectum. Draw LM perpendicular to the directrix.

Since the parabola is symmetrical, with respect to the axis, $LS = L'S$. Therefore

$$LL' = 2LS = 2LM = 2XS = 4AS.$$

Ex. 1. Find a double ordinate of a parabola which shall be double the latus rectum.

Ex. 2. The radius of the circle described about the triangle $LAL' = \frac{5}{8}$ latus rectum.

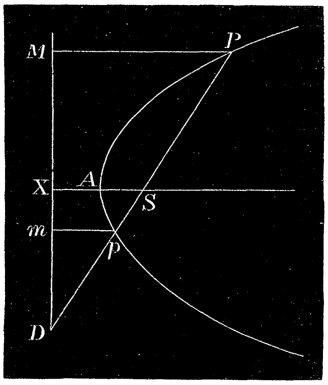
Ex. 3. Find the point O in a given ordinate PN, such that OR being drawn parallel to the axis to meet the curve in R, $ON + OR$ may be the greatest possible. $[ON = 2AS.]$

*Proposition III.

Any focal chord of a parabola is divided harmonically by the curve, the focus, and the directrix.

Def. A straight line AB is said to be divided harmonically in O and O', if it is divided internally in O and externally in O', in the same ratio, that is, if

$$AO : OB = AO' : O'B.$$



Produce the focal chord PSp to meet the directrix in D, and draw PM, pm from P, p, perpendicular to the directrix.

Then, from the similar triangles DMP, Dmp,

$$PD : pD = PM : pm.$$

But $PM = PS$, and $pm = pS$.

Therefore $$PD : pD = PS : pS.$$

Hence Pp is divided harmonically in S and D.

Ex. 1. Prove that $\frac{1}{PS}+\frac{1}{PD}=\frac{2}{Pp}$.

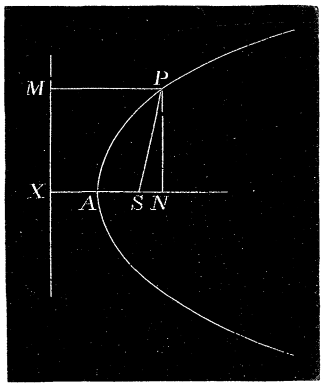
Ex. 2. Prove that $\frac{1}{DP}+\frac{1}{Dp}=\frac{2}{DS}$.

Ex. 3. The semi-latus rectum is a harmonic means between the two segments of any focal chord of a parabola.

Ex. 4. Focal chords of a parabola are to one another as the rectangles contained by their segments.

PROPOSITION IV.

The square of the ordinate of any point on a parabola is equal to the rectangle contained by the latus rectum and the abscissa ($PN^2=4AS.AN$).



Draw PM perpendicular to the directrix, and join SP. Then, because XS is bisected in A and produced to N,

$$NX^2=SN^2+4AS.AN. \quad \text{[Euc. II. 8.}$$

But $NX=PM=SP$.

Therefore $NX^2=SP^2=SN^2+PN^2$. [Euc. I. 47.

Therefore $PN^2=4AS.AN$.

Ex. 1. If PL be drawn at right angles to AP, meeting the axis in L, NL is always equal to the latus rectum.

Ex. 2. If a circle be described about the triangle SPN, the tangent to it from $A = \frac{1}{2}PN$.

Ex. 3. A straight line parallel to the axis bisects PN, and meets the curve in Q; NQ meets a line through A at right angles to the axis, in T. Prove that $3AT = 2 . PN$.

Ex. 4. If SQ be parallel to AP, and QM be the ordinate of Q, prove that $SM^2 = AM . AN$.

Ex. 5. If O be any point on a double ordinate PNP', and OQ parallel to the axis meets the curve in Q, show that

(i.) $OP . OP = 4AS . OQ$;
(ii.) $PN : ON = OR : QR$.

Ex. 6. PNP' is a double ordinate of a parabola. Through Q, another point on the curve, straight lines are drawn, one passing through the vertex, the other parallel to the axis, cutting PP' in l, l'. Prove that $PN^2 = Nl . Nl'$.

Ex. 7. A circle has its centre at A, and its diameter is equal to $3AS$. Show that the common chord of the circle and the parabola bisects AS,

Ex. 8. AP, BQ are two lines at right angles to AB; A is joined to any point Q on BQ; a point O is taken on AQ such that the perpendicular ON on $AP = BQ$. Prove that the locus of O is a parabola. [Axis, AP; Latus rectum, AB.]

Ex. 9. PM, QN are the ordinates of the extremities of two chords AP, AQ which are at right angles to each other. Prove that $AM . AN = (\text{Latus rectum})^2$.

Ex. 10. The latus rectum is a mean proportional between the double ordinates of the extremities of a focal chord. (See Prop. I., Ex. 16).

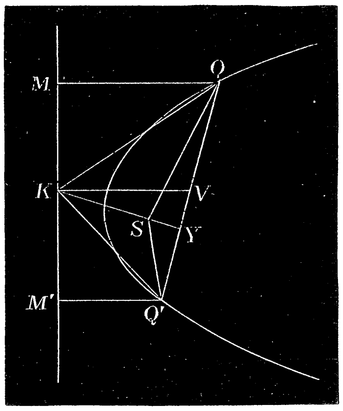
Ex. 11. PSp is a focal chord; prove that AP, Ap meet the latus rectum in points whose focal distances are equal to the ordinates of p and P respectively. (Apply Prop. I., Ex. 16.)

PROPOSITION V.

The locus of the middle points of any system of parallel chords of a parabola is a straight line parallel to the axis.

Let QQ' be one of a system of parallel chords. Draw QM, $Q'M'$ perpendicular to the directrix. Draw SY

perpendicular to QQ', produce YS to meet the directrix in K, and draw KV parallel to the axis. Then KV shall bisect QQ'. Join KQ, KQ', SQ, and SQ'.



Then $MK^2 = KQ^2 - MQ^2$ [Euc. I. 47

$= KQ^2 - QS^2$.

But $KQ^2 = KY^2 + QY^2$ [Euc. I. 47.

and $QS^2 = SY^2 + QY^2$. [Euc. I. 47.

Therefore $MK^2 = KY^2 - SY^2$.

Similarly $M'K^2 = KQ'^2 - M'Q'^2$

$= KQ'^2 - Q'S^2$

$= KY^2 - SY^2$.

Therefore $MK = M'K$,

but, since KV is parallel to MQ and $M'Q'$, QQ' is bisected at V.

Now QQ' being fixed in direction and KSY being perpendicular to it, KSY is a fixed straight line and K is a fixed point. Therefore KV, which is parallel to the axis, is a fixed straight line bisecting all chords parallel to QQ'.

Def. A *diameter* of any curve is the locus of the middle points of a system of parallel chords drawn in the curve.

It has just been proved that the diameters of a parabola are straight lines. It will be shown hereafter that the diameters of the other conics are also straight lines. It should be observed, however, that a diameter is not necessarily a straight line for all curves.

Def. The half chords (QV, $Q'V$) intercepted between the diameter and the curve, are called the *ordinates* to the diameter.

Def. The *abscissa* of a point on a parabola with respect to any diameter is the portion of the diameter intercepted between the ordinate of the point and the parabola.

Def. In the parabola, the *vertex* of a diameter is the point in which it cuts the curve.

Ex. 1. The perpendicular from the focus upon a system of parallel chords intersects the diameter bisecting the chords upon the directrix.

Ex. 2. If a system of parallel chords make an angle of 45° with the axis, their diameter passes through an extremity of the latus rectum (see Prop. IV.).

Ex. 3. A parabola being traced on paper, find its focus and directrix.

The direction of the axis is given by the straight line joining the middle points of a pair of parallel chords. The position of the axis is found by observing that the middle point of any chord at right angles to its direction lies on it. At any point N on the axis, draw a perpendicular to it $NK=2AN$. Join KA, cutting the curve in L, which will be an extremity of the latus rectum.

Ex. 4. The difference between the segments of any focal chord is equal to the parallel chord through the vertex.

Ex. 5. QSQ' is a focal chord; QM, $Q'M'$ are perpendicular to the axis. Show that MM' is equal to the parallel chord through the vertex.

Ex. 6. AP is any chord through the vertex, and PE is drawn at right angles to AP, meeting the axis in E. AE is equal to the focal chord parallel to AP.

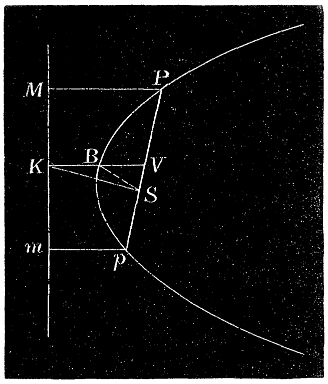
Ex. 7. The middle points of any two chords of a parabola equally inclined to the axis, are equidistant from the axis.

Ex. 8. If a parabola drawn through the middle points of the sides of a triangle ABC meets the sides again in α, β, γ, the lines $A\alpha$, $B\beta$, $C\gamma$ will be parallel to each other. [Each is parallel to the axis.]

PROPOSITION VI.

The parameter of any diameter of a parabola is four times the line joining the focus with the vertex of the diameter.

Def. The *parameter* of a diameter is the length of the focal chord bisected by the diameter.



Draw SK at right angles to the focal chord PSp, to meet the directrix in K; draw PM, pm at right angles to the directrix, and KBV parallel to them. Then KBV is the diameter bisecting the chord PSp (Prop. V.). Join SB.

Then, since KSV is a right angle, and $KB=BS$, we have

$$KB=BS=BV,$$

or

$$KV=2BS.$$

Now, because Pp is bisected in V,

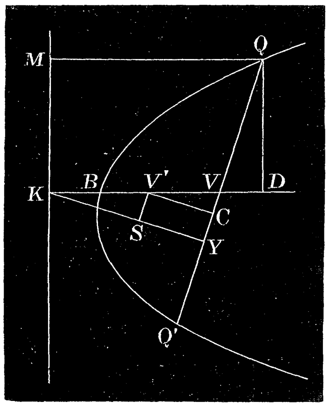
$$Pp=PS+Sp=PM+pm$$
$$=2KV=4BS.$$

Ex. 1. Given the length of a focal chord, find its position.

Ex. 2. Draw a focal chord PSp, such that $SP=3Sp$.

Proposition VII.

The ordinate to any diameter of a parabola at any point is a mean proportional to its parameter and the abscissa of the point with respect to the diameter ($QV^2=4BS.BV$).



Let QQ' be any chord. Draw SY at right angles to it, and produce YS to meet the directrix in K. Draw KBV

parallel to the axis, so that BV is the diameter bisecting QQ' in V, QV being the ordinate and BV the abscissa. [Prop. V.

Draw SV' parallel to QQ', and QM, QD, $V'C$ at right angles to the directrix, KV and QQ' respectively.

Then $QD^2 = MK^2$

$= KY^2 - SY^2$; [Prop. V.

and, from the similar triangles QVD, KVY, and $V'VC$,

$$QD : QV = KY : KV$$
$$= V'C : V'V$$
$$= SY : V'V.$$

Therefore $QV^2 = KV^2 - V'V^2$.

But as KV' is bisected in B, [Prop. VI.

$KV^2 = V'V^2 + 4BV \,.\, BV'$. [Euc. II. 8.

Therefore $QV^2 = 4BV \,.\, BV'$

$= 4BS \,.\, BV$. [Prop. VI.

Ex. 1. If any chord BR meets QM and QQ' in L and N, prove that $BL^2 = BN \,.\, BR$.

Ex. 2. If QQ' meets any chord BR in N, and the diameter through R in N', prove that $QV^2 = VN \,.\, VN'$.

Ex. 3. If QOQ' be any chord meeting the diameter BV in O, and QV, $Q'V'$ ordinates to the diameter, then $BO^2 = BV \,.\, BV'$.

Let QB produced meet the diameter through Q' in E, and draw ER parallel to the ordinate meeting BV produced in R.

Then $QV^2 : Q'V'^2 = BV^2 : BV \,.\, BV'$.

But $QV^2 : BV^2 = Q'V'^2 : BR^2$;

∴ $BV \,.\, BV' = BR^2$;

∴ $BV : BR = BR : BV'$;

or $BV : RV = BR : RV'$.

But $BV : RV = QB : QE$

$= BO : RV'$;

∴ $BO = BR$.

Ex. 4. If POP' be the chord bisected by the diameter BOV at O, $PO^2 = QV \,.\, Q'V'$.

Ex. 5. Through a given point, to draw a chord of a parabola which will be divided in a given ratio at the point.

Through the given point O, draw the diameter BO. Then if V, V' be the feet of the ordinates drawn through the extremities of the chord sought, it is clear that $BV' : BV$ is as the square of the given ratio. Also, $BV.BV'=BO^2$, whence the points V, V' are known.

Ex. 6. If any diameter intersect two parallel chords, the rectangles under the segments of these chords are proportional to the segments of the diameter intercepted between the chords and the curve.

If QQ' be one of the chords meeting the diameter BV in V, and if O be its middle point,

$$QV.Q'V=QO^2-OV^2=4BS.BV.$$

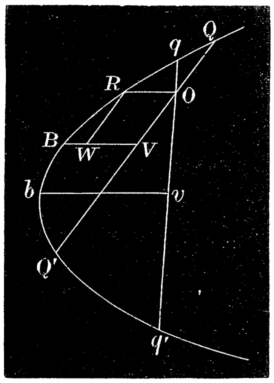
Ex. 7. QQ' is a fixed straight line, and from any point V in it, VB is drawn in a fixed direction such that BV is proportional to $QV.Q'V$. Show that the locus of B is a parabola passing through Q, Q' and having its axis parallel to BV.

Ex. 8. Given the base and area of a triangle, the locus of its orthocentre is a parabola.

Ex. 9. BO, $B'O'$ are any two diameters. A line is drawn parallel to the ordinate to BO, cutting the curve in D, and BO, BB', $B'O$ in O, C, E respectively. Prove that $OD^2=OC.OE$. (Through B' draw a parallel to EO.)

*Proposition VIII.

If two chords of a parabola intersect each other, the rectangles contained by their segments are in the ratio of the parallel focal chords.



Let the chords QQ' and qq' intersect in a point O

within the parabola. Bisect QQ' in V, and draw the diameters OR, VB. Draw RW parallel to QQ'.

Then, because QQ' is bisected in V,

$$QO\,.\,Q'O = QV^2 - OV^2 \qquad \text{[Euc. II. 5.}$$
$$= QV^2 - RW^2$$
$$= 4BV\,.\,BS - 4BW\,.\,BS \qquad \text{[Prop. VII.}$$
$$= 4BS\,.\,WV$$
$$= 4BS\,.\,OR.$$

Similarly, if bv be the diameter bisecting qq',

$$qO\,.\,q'O = 4bS\,.\,OR.$$

Therefore $\quad QO\,.\,Q'O : qO\,.\,q'O = 4BS : 4bS$;

that is, as the focal chords parallel to QQ' and qq' respectively. [Prop. VI.

The proposition may be similarly proved when the chords intersect outside the curve.

Ex. 1. If two intersecting chords be parallel to two others, the rectangles contained by the segments of the one pair are proportional to the rectangles contained by the segments of the other pair.

Ex. 2. Deduce Prop. III.

Ex. 3. Given three points on a parabola and the direction of the axis, construct the curve.

Ex. 4. Inscribe in a given parabola a triangle having its sides parallel to three given straight lines.

*Proposition IX.

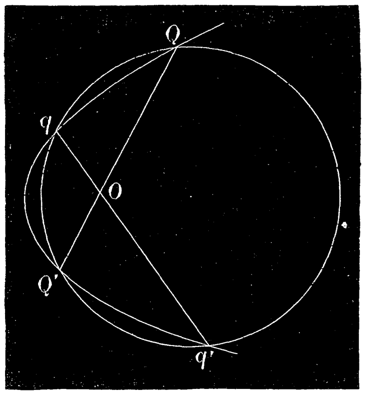
If a circle intersect a parabola in four points their common chords will be equally inclined, two and two, to the axis.

Let Q, Q', q, q' be the four points of intersection.

Then $\qquad QO\,.\,Q'O = qO\,.\,q'O.$ [Euc. III. 35.

Therefore, the focal chords parallel to QQ' and qq' are equal to each other. [Prop. VIII.

And they are therefore equally inclined to the axis, from the symmetry of the figure. (See also Prop. I., Ex. 10.)



Therefore the chords QQ', qq' are equally inclined to the axis.

In like manner, it may be shown that the chords Qq and $q'Q'$, as well as the chords Qq' and qQ', are equally inclined to the axis.

Ex. 1. If a circle cut a parabola in four points, two on one side of the axis and two on the other, the sum of the ordinates of the first two is equal to the sum of the ordinates of the other two points. (See Prop. V., Ex. 7.)

Ex. 2. If three of the points are on the same side of the axis, the sum of their ordinates is equal to the ordinate of the fourth point.

Proposition X.

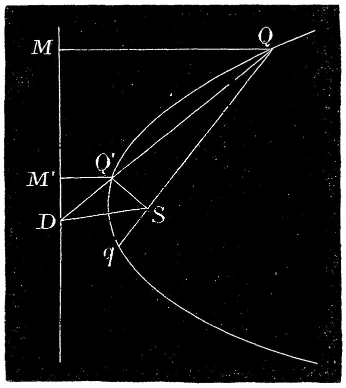
If any chord QQ' of a parabola intersects the directrix in D, SD bisects the exterior angle between SQ and SQ'.

Draw QM, $Q'M'$ perpendicular to the directrix.

Then, by similar triangles,

$$\begin{aligned} QD : Q'D &= QM : Q'M' \\ &= SQ : SQ'. \end{aligned}$$

Therefore SD bisects the exterior angle $Q'Sq$. [Euc. VI. A.



Ex. 1. Given the focus and two points on a parabola, find the directrix.

The point D, being the intersection of the chord QQ' and the bisector of the angle $Q'Sq$, is on the directrix, which touches the circle described with Q as centre and radius QS.

Ex. 2. PQ, pq are focal chords. Show that Pp, Qq, as also Pq, pQ, meet on the directrix.

If they meet the directrix in K, K', KSK' is a right angle.

Ex. 3. Given the focus and the directrix, trace the parabola by means of this proposition. (For other constructions, see Prop. I., and Ex. 2, Ex. 3.)

Determine the vertex A as the middle point of SX. Take any point D on the directrix; make the angle DSp equal to the angle DSA, and let pS and DA produced meet in P. P is a point on the parabola.

Ex. 4. Q is a point on the parabola. If QA produced meet the directrix in D, MSD is a right angle.

Ex. 5. PQ is a double ordinate, and PX cuts the curve in P'; show that the focus lies on $P'Q$.

Ex. 6. If two fixed points Q, Q' on a parabola be joined with a third variable point O on the curve, the segment qq' intercepted on the directrix by the chords QO, $Q'O$ produced, subtends a constant angle at the focus.

The angle qSq' may be proved to be equal to half of the angle QSQ'.

Ex. 7. If QQ' be a focal chord, the angle qSq' is a right angle, and $qX \,.\, q'X = (\text{semi-latus rectum})^2$.

Ex. 8. Show that a straight line which meets a parabola will, in general, meet it in two points, except when the line is parallel to the axis, in which case it meets the curve in one point only; and no straight line can meet the curve in more points than two.

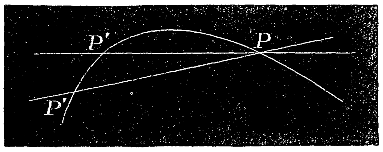
Let DQ' be any straight line which meets the directrix in D and the curve in Q'. Make the angle DSq equal to the angle DSQ', and let qS, DQ' intersect in Q. Then since

$$SQ : SQ' = QD : Q'D = QM : Q'M,$$

Q is a point on the curve. If, however, DQ' be parallel to the axis, qS will coincide with the axis, and $D'Q'$ will meet the parabola in the point Q' only (the other point of intersection in this case being really at infinity). Again SQ, SQ', being equally inclined to DS, if there be a third point of intersection Q'', SQ, SQ'' will make the same angle with DS, which is impossible.

PROPERTIES OF TANGENTS.

Def. A *tangent* to a conic is the limiting position of a chord whose two points of intersection with the curve have become coincident.



Thus, if P and P' be two points on a conic, and if the chord PP' be so turned about P that P' may approach P, then in the limiting position when P' moves up to P and coincides with it, the chord becomes the *tangent* to the conic at P.

Again, if a chord PP' moves parallel to itself until P and P' coincide at a point B on the conic, PP' becomes in its limiting position the tangent to the curve at the point B.

Hence, a tangent may be said to be a straight line which passes through two *consecutive* or *coincident* points on the curve.

It will be seen that, generally, to a chord-property of a conic, there corresponds a tangent-property.

Thus, in Prop. V., if the chord QQ' moves parallel to itself until Q' coincides with Q at the point B on the curve, the chord in this its limiting position becomes the tangent to the parabola at B, which is thus seen to be parallel to the system of chords bisected by the diameter BV. (See Prop. XI.)

* Again, in Prop. VIII., let the chords QQ', qq' intersect at a point O *outside* the parabola. Let the chord OQQ' be made to turn about the point O, until Q' coincides with Q at a point R on the curve, so that OR becomes the tangent to the curve at the point R, and OQ, OQ' become each equal to OR. In like manner, let Oqq' be made to turn about the point O, until q' coincides with q at a point r on the curve, so that Or becomes the tangent to the curve at the point r, and Oq, Oq', become each equal to Or. Hence, we have the following proposition :—

The squares of any two intersecting tangents to a parabola are in the ratio of the parallel focal chords.

Ex. 1. If OTO' be the tangent to a parabola at T, and if OPQ, $O'P'Q'$ be a pair of parallel chords,

$$OT^2 : O'T^2 = OP \,.\, OQ : O'P' \,.\, O'Q'.$$

Ex. 2. If TOO' be the tangent to a parabola at T, $O'P'$ a tangent from O', and OPQ a chord parallel to $O'P'$, cutting the chord of contact $P'Q$ in R, prove that $OP \,.\, OQ = OR^2$.

From Ex. 1,

$$OP \,.\, OQ : OT^2 = O'P'^2 : O'T^2 = OR^2 : OT^2.$$

Cf. Prop. XXI., Ex. 8.

* Next, in Prop. IX., suppose q to coincide with Q, and

therefore also with O; then the circle and the parabola will touch each other at O, the chords OQ', oq' being equally inclined to the axis. Hence

If two chords OP, OQ of a parabola are equally inclined to the axis, the circle round OPQ touches the parabola at O.

Ex. If one of the chords OP be at right angles to the tangent to the curve at O, the angle OQP is a right angle.

Similarly, if a circle touches a parabola at O and cuts it again in P and Q, the tangent at O and PQ are equally inclined to the axis.

Ex. If a circle touches a parabola at O and cuts it in P and Q, and PU, QV parallel to the axis meet the circle in U, V, show that UV is parallel to the tangent at O.

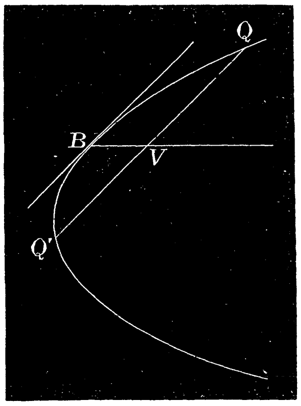
Again, consider Prop. X. Let the chord QQ' be made to turn about Q, until Q' coincides with Q, so that the chord becomes the tangent to the parabola at the point Q. The angle QSQ' vanishes, and, therefore, the exterior angle $Q'Sq$ becomes equal to two right angles. But since SD always bisects the angle $Q'Sq$, SD will, in this limiting position, be at right angles to SQ. Hence the following proposition:—

The tangent to a parabola from any point on the directrix, subtends a right angle at the focus. (See Prop. XII.)

Def. A circle or a conic is said to *touch* a conic at a point P when they have a *common tangent* at that point.

Proposition XI.

The tangent to a parabola at its point of intersection *with a diameter is parallel to the system of chords bisected by the diameter.*



Let BV be the diameter bisecting a system of chords parallel to QQ'.

Let QQ' be made to move parallel to itself, so that Q may coincide with V. Since QV is always equal to $Q'V$ (Prop. V.), it is clear that Q' will also coincide with B, or, the chord in this, its limiting position, will be the tangent to the parabola at B.

Ex. Draw a tangent to a parabola making a given angle with the axis.

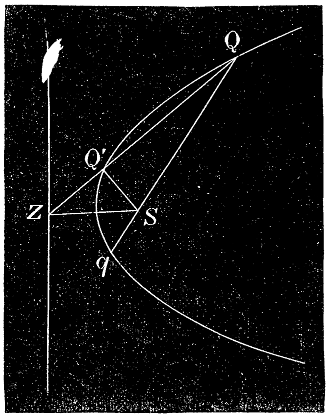
Proposition XII.

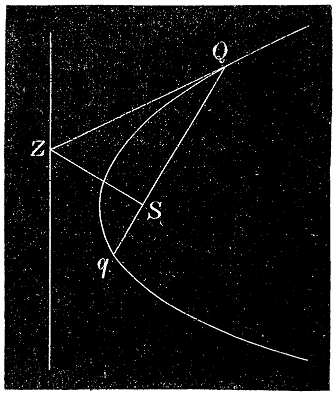
The portion of the tangent to a parabola at any point, intercepted between that point and the directrix, subtends a right angle at the focus.

Let any chord QQ' of the parabola intersect the directrix in Z.

Then SZ bisects the exterior angle $Q'Sq$. [Prop. X.

Now, let the chord QQ' be made to turn about Q until the point Q' moves up to and coincides with Q, so that the chord becomes the tangent to the parabola at Q. In this limiting position of the chord QQ', since Q and Q' coincide, the angle QSQ' vanishes, and therefore the angle $Q'Sq$ becomes equal to two right angles. But since





SZ always bisects the angle $Q'Sq$, in this case the angle QSZ is a right angle.

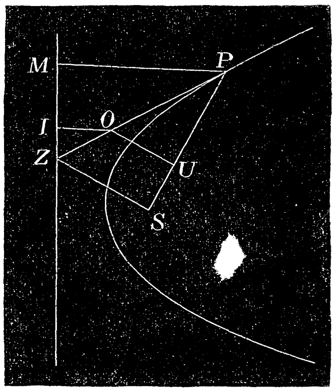
Ex. 1. If a line QZ meeting the curve in Q and the directrix in Z, subtend a right angle at the focus, it will be the tangent to the curve at Q.

Ex. 2. The tangents at the extremities of the latus rectum meet the directrix on the axis produced.

* PROPOSITION XIII.

If from any point O on the tangent at P of a parabola perpendiculars OU and OI be drawn to SP and the directrix respectively, then

$$SU = OI.$$



Join SZ, and draw PM perpendicular to the directrix.

Because ZSP is a right angle, [Prop. XII.

ZS is parallel to OU.

Therefore, by similar triangles,

$$SU : SP = ZO : ZP$$
$$= OI : PM.$$

But $SP = PM$;

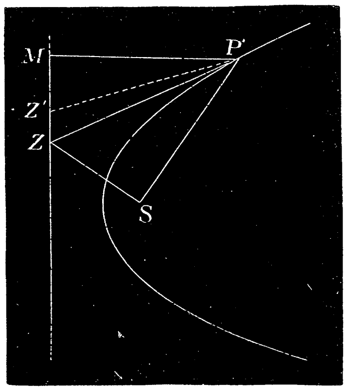
therefore $SU = OI$.

This property of the parabola is the particular case of a general property of all conics discovered by Adams.

Ex. If a line OP meet the parabola at P, and OI, OU being drawn at right angles to the directrix and SP respectively, $SU=OI$, then OP will be the tangent to the curve at P.

PROPOSITION XIV.

The tangent at any point of a parabola bisects the angle which the focal distance of the point makes with the perpendicular drawn from the point on the directrix, and conversely.



Let the tangent at the point P meet the directrix in Z. Draw PM perpendicular to the directrix, and join SP, SZ.

Then, since the angle PSZ is a right angle, [Prop. XII.

$$SP^2+SZ^2=PZ^2. \qquad \text{[Euc. I. 47.}$$

Also $$PM^2+MZ^2=PZ^2; \qquad \text{[Euc. I. 47.}$$

therefore $$SP^2+SZ^2=PM^2+MZ^2.$$

But $$SP=PM;$$

therefore $$SZ=MZ.$$

Now, in the two triangles ZPM, ZPS, the two sides PM, MZ are respectively equal to the two sides SP, SZ, and the side PZ is common; therefore the two triangles are equal, and the angle SPZ is equal to the angle MPZ, that is, PZ bisects the angle SPM.

Conversely, if PZ bisects the angle SPM, PZ is the tangent at P. For, if not, and if possible, let any other line PZ' be the tangent at P, then by what has been proved PZ' will bisect the angle SPM, which is impossible; therefore PZ is the tangent at P.

Note.—It may be shown from the definition of the parabola that the straight line which bisects the angle between SP and PM cannot meet the curve again in any other point; hence PZ would also be the tangent to the parabola at P, according to Euclid's definition of a tangent.

Corollary.—The tangent at the vertex of a parabola is at right angles to the axis.

Ex. 1. Show how to draw the tangent at a given point of a parabola.

Ex. 2. Draw a tangent to a parabola making a given angle with the axis.

Ex. 3. If the tangent at P meets the axis in T, $SP=ST$.

Ex. 4. Two parabolas have the same focus, and their axes in the same straight line, but in opposite directions. Prove that they intersect at right angles.

Note.—Two curves are said to intersect at right angles when their tangents at a common point are at right angles.

Ex. 5. Given the vertex of a diameter of a parabola and a corresponding double ordinate, construct the curve. (Apply Prop. VII.)

Ex. 6. If ZP be produced to R, the angles SPR and MPR are equal.

Ex. 7. PZ bisects SM at right angles.

Ex. 8. Any point O on the tangent at P is equidistant from M and S.

Ex. 9. If the tangents to the parabola at Q and Q' meet in O, and QM, $Q'M'$ be the perpendiculars on the directrix from Q and Q', OM, OS, OM' are all equal.

Hence deduce, by analysis, the construction for Prop. XVII., namely, to draw two tangents to a parabola from an external point O.

Ex. 10. The tangent at any point of a parabola meets the directrix and the latus rectum in two points equidistant from the focus.

Ex. 11. The focal distance of any point on a parabola is equal to the length of the ordinate of that point produced to meet the tangent at the end of the latus rectum. (See Prop. XII., Ex. 2.)

Ex. 12. O is a point on the tangent at P, such that the perpendicular from O on SP is equal to $2AS$; find the locus of O. (A parabola of which the vertex is on the directrix of the given one. Apply Prop. VII., Ex. 7.)

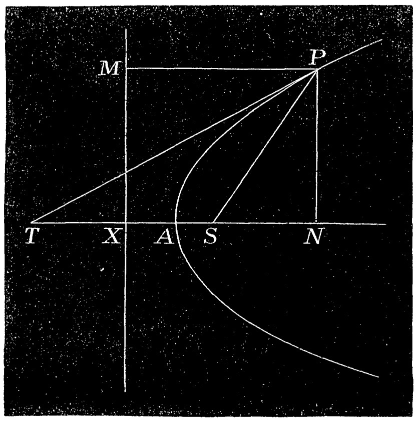
Ex. 13. If a leaf of a book be folded so that one corner moves along an opposite side, the line of the crease touches a parabola.

Let the leaf $BCXS$ be so folded that S coincides with a point M on CX; let the crease TT' meet XS, BS in T, T' respectively. Draw MP at right angles to CX, meeting TT' in P; join SP. Then $SP=PM$, $\angle SPT=\angle MPT$; TT', therefore, touches at P a parabola, of which the focus is S and directrix C.

Def. The portion of the axis intercepted between the tangent at any point of a conic and the ordinate of that point is called the *subtangent*.

* Proposition XV.

The subtangent of any point of a parabola is bisected at the vertex, that is, is equal to double the abscissa of the point with respect to the axis.



Let the tangent PT at P meet the axis in T. Draw

PN, PM perpendicular to the axis and directrix respectively.

Then, the angle STP = the angle TPM

= the angle TPS. [Prop. XIV.

Therefore $ST = SP = PM = XN$.

But $AS = AX$.

Therefore $AT = AN$,

or $NT = 2AN$.

Ex. 1. If T is the middle point of AX, prove that N is the middle point of AS.

Ex. 2. The radius of the circle described round the triangle TPN is $\sqrt{(SP \cdot AN)}$.

Ex. 3. The locus of the middle points of the focal chords of a parabola is another parabola having the same axis and passing through the focus. (Apply Prop. VII., Ex. 7.)

Ex. 4. The diameter through P meets at E, a right line through S parallel to the tangent at P. Prove that the locus of E is a parabola.

If En be perpendicular to the axis, $nS = NT = 2AN$. If S' be taken on the axis, such that $2SS' = AS$, the relation $PN^2 = 4AS \cdot AN$ gives $En^2 = 4SS' \cdot Sn$, showing the locus to be a parabola whose axis coincides with that of the original one, whose vertex is at S, and latus rectum half that of the original parabola.

Ex. 5. If SM meets PT in Y, $NY = TY$.

Ex. 6. If the tangent at P meets the tangent at the vertex in Y, $AY^2 = AS \cdot AN$.

Ex. 7. If SE be the perpendicular from S on the line through P at right angles to PT, show that $SE^2 = AN \cdot SP$. ($2SE = PT$. Apply Prop. IV.)

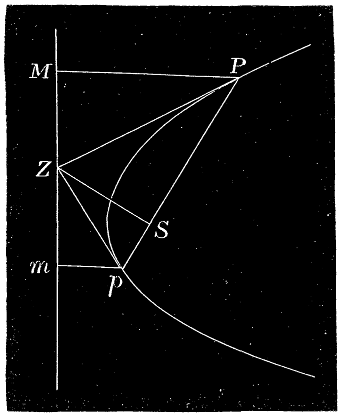
Ex. 8. Given the vertex, a tangent and its point of contact, construct the curve.

Produce PA to P', such that $AP' = AP$; if the circle on AP' as diameter meets the tangent at P in T, TA is the axis. Then apply Prop. XIV.

Ex. 9. Find the locus of the intersection of the perpendicular from the vertex on the tangent at any point with the diameter through that point. (A right line parallel to the directrix. Apply Prop. IV.)

* Proposition XVI.

The tangents at the extremities of a focal chord of a parabola intersect at right angles on the directrix.



Draw SZ at right angles to the focal chord PSp, meeting the directrix in Z. Join PZ, pZ, and draw PM, pm perpendiculars to the directrix.

Then $ZP^2 = ZS^2 + SP^2$

$= ZM^2 + PM^2$. [Euc. I. 47.

But $SP = PM$.

Therefore $ZS = ZM$.

Therefore from the triangles ZSP and ZMP, the angle SPZ = the angle MPZ, and the angle SZP = the angle MZP. [Euc. I. 8.

Similarly,

the angle SpZ = the angle mpZ,

and the angle SZp = the angle mZp.

Therefore, PZ and pZ are the tangents at P and p. [Prop. XIV.

Also,

$$\text{the angle } PZp = \tfrac{1}{2} \text{ the angle } MZS + \tfrac{1}{2} \text{ the angle } mZS$$
$$= \text{one right angle}$$

Ex. 1. Show that Mm is bisected in Z.

Ex. 2. If two tangents be drawn to a parabola from any point on the directrix, they shall be at right angles.

Ex. 3. If perpendiculars through P, p, to ZP, Zp respectively, meet in O, the distance of O from the directrix varies as $PS.pS$. (Apply Prop. III., Ex. 4.)

Ex. 4. Find the locus of O in Ex. 3. [A parabola having the same axis as the given one.]

Ex. 5. Show that the circle described on the focal chord Pp as diameter touches the directrix at Z.

Ex. 6. If a circle described upon a chord of a parabola as diameter meets the directrix, it also touches it; and all chords for which this is possible, intersect in a fixed point. [The focus.]

The distance of the middle point of the chord from the directrix is always greater than half the chord, unless the chord passes through the focus.

Ex. 7. Tangents at the extremities of a focal chord cut off equal intercepts on the latus rectum. (Apply Prop. XIV., Ex. 10.)

Ex. 8. Prove that SM, Sm are respectively parallel to Zp, ZP.

Ex. 9. The locus of the intersection of any two tangents to a parabola at right angles to each other, is the directrix.

Ex. 10. Given two tangents at right angles, and their points of contact, construct the curve.

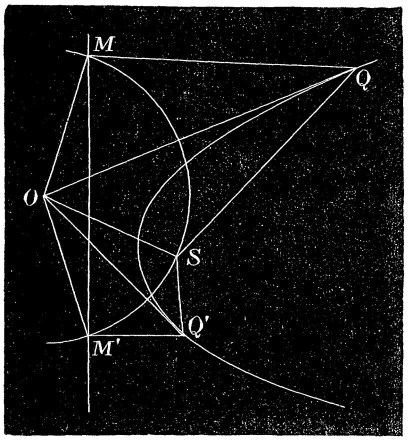
PROPOSITION XVII.

To draw two tangents to a parabola from an external point.

Let O be the external point. With centre O and radius OS, describe a circle cutting the directrix in M and M'. Draw MQ, $M'Q'$ at right angles to the directrix to meet the parabola in Q and Q'. Join OQ and OQ'; these shall be the tangents required.

Join OS, OM, OM' SQ and SQ'.

Then, in the triangles OQM, OQS, the sides MQ, QO are equal to the sides SQ, QO respectively, and OM is equal to OS. Therefore the angles OQM, OQS are equal. Therefore OQ is the tangent to the parabola at Q.



[Prop. XIV.

Similarly, OQ' is the tangent at Q'.

Note.—For an analysis of the construction, see Prop. XIV., Ex. 9.

It should be observed that in order that the construction may be possible, the circle described with O as centre and with radius OS must meet the directrix, that is, the distance of O from S must be either greater than or equal to its distance from the directrix. The former is the case when the point is outside the parabola (Prop. I., Ex. 7); and as in this case the circle must intersect the directrix in two points only, it follows that two tangents, and no more, can be drawn to a parabola from an *external* point. In the second case the point O is evidently on the parabola, and the circle touches the directrix, that is, meets it in two coincident points; the two tangents in this case coincide, that is, only one tangent can be drawn to a parabola at a given point on it. The distance of any point inside the parabola being less than its distance from the directrix (Prop. I., Ex. 6), no tangent can be drawn to a parabola from any point within it.

Ex. 1. If the point O be on the directrix, show from the construction that the tangents intersect at right angles.

Ex. 2. If O be on the axis produced, at a distance from the vertex $A = \frac{1}{3}AS$, the figure $OQSQ'$ will be a rhombus.

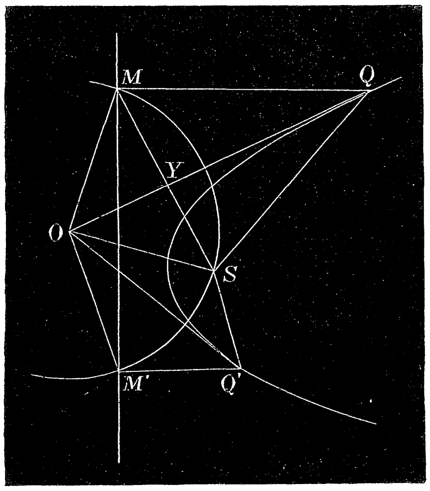
Ex. 3. *Alternative Construction.*—With the given point O as centre and radius OS, describe a circle cutting the directrix in M and M'. The perpendiculars from O upon SM and SM' will, when produced, touch the curve. (See Prop. I., Ex. 3.)

Ex. 4. *Alternative Construction.*—In the figure of Prop. XIII., taking O as the given point, draw OI at right angles to the directrix. With centre S and radius equal to OI, describe a circle; and from O draw OU and OU' tangents to this circle. SU, SU' produced will meet the parabola in the points of contact of the tangents from O. (See Prop. XIII., Ex.)

For another alternative construction, see Prop. XXIII., Ex. 13.

PROPOSITION XVIII.

The two tangents OQ, OQ' of a parabola subtend equal angles at the focus; and the triangles SOQ, SOQ' are similar.



With centre O and radius OS, describe a circle cutting the directrix in M and M'; draw MQ, $M'Q'$ at right angles to the directrix to meet the curve in Q, Q'. Then OQ and OQ' are the tangents to the curve from O. [Prop. XVII.

Join OM, OM', OS, SQ, SQ', and SM, cutting OQ in Y.

In the two triangles MQY and SQY, the sides MQ, QY are equal to the sides SQ, QY, and the angles MQY, SQY are equal; [Prop. XIV.
therefore the two triangles are equal in every respect; and the angles MYQ, SYQ are equal, each being thus equal to a right angle. [Euc. I. 4.

Now, the angle $SQO =$ the angle MQO,
and the angle $MQO =$ the angle SMM',
each being the complement of the angle QMY.

Therefore
the angle $SQO =$ the angle SMM'.
But the angle $SMM' = \frac{1}{2}$ the angle SOM', [Euc. III. 20.
and from the equality of the triangles SOQ', $M'OQ'$, [Prop. XVII.
the angle $SOQ' =$ the angle $M'OQ'$,
or, the angle $SOQ' = \frac{1}{2}$ the angle SOM'.
Therefore the angle $SQO =$ the angle SOQ'.

Similarly, the angles QOS and $OQ'S$ are equal, as also the remaining angles QSO, $Q'SO$.

Therefore the two triangles SOQ, SOQ' are similar.

Ex. 1. Prove that
(i.) $SQ \,.\, SQ' = SO^2$; (ii.) $OQ^2 : OQ'^2 = SQ : SQ'$.

Ex. 2. If two tangents drawn from any point on the axis be cut by any third tangent, the points of intersection are equidistant from the focus.

Ex. 3. The angle subtended at the focus by the segment intercepted on a variable tangent by two fixed tangents, is constant.

Ex. 4. OS and a line through O parallel to the axis make equal angles with the tangents.

Ex. 5. The straight line bisecting the angle QOQ' meets the axis in R; prove that $SO=SR$.

Ex. 6. If two tangents drawn from any point on the axis be cut by a third tangent, their alternate segments are equal. (Cf. Prop. XXI., Ex. 10.)

Ex. 7. If the tangent and normal at any point P of a parabola meet the tangent at the vertex in K and L respectively, prove that

$$KL^2 : SP^2 = SP - AS : AS.$$

Ex. 8. If from any point on a given tangent to a parabola, tangents be drawn to the curve, the angles which these tangents make with the focal distances of the points from which they are drawn, are all equal.

Each angle is equal to the angle between the given tangent and the focal distance of the point of contact.

Ex. 9. Of the two tangents drawn to a parabola from any point, one makes with the axis the same angle as the other makes with the focal distance of the point.

Ex. 10. Two parabolas have the same focus and axis, with their vertices on the same side of their common focus. Tangents are drawn from any point P on the outer parabola to the inner one. Show that they are equally inclined to the tangent at P to the outer curve. (Apply Ex. 9, and Prop. XIV.)

Ex. 11. If the tangent at any point R meets OQ, OQ' in q, q', show that $Qq : qO = Oq' : q'Q' = qR : Rq'$.

[The triangles OqS, $Rq'S$ are similar.]

Ex. 12. If tangents be drawn from any point on the latus rectum, show that the semi-latus-rectum is a geometric mean between the ordinates of the points of contact. (Apply Prop. I., Ex. 16, and Prop. IV.)

Ex. 13. If PV, $P'V'$ be two diameters, and $P'V$, PV' ordinates to these diameters, show that $PV=P'V'$. (Apply Prop. VII. and Ex. 1.)

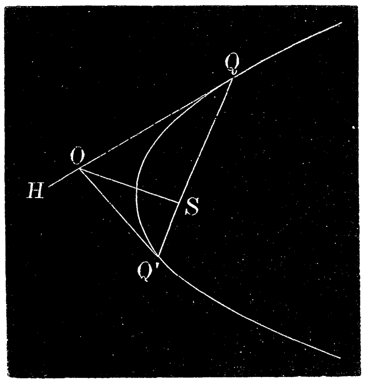
Ex. 14. If one side of a triangle be parallel to the axis of a parabola, the other sides will be in the ratio of the tangents parallel to them.

Proposition XIX.

The exterior angle between any two tangents to a parabola is equal to the angle which either of them subtends at the focus.

Let OQ and OQ' be the two tangents, and S the focus. Join SO, SQ, and SQ'.

The angle SOQ' = the angle SQO. [Prop. XVIII. To each of these equals add the angle SOQ; therefore the angles SOQ and SQO are together equal to the angle QOQ'. But the exterior angle HOQ' is the supplement of the angle QOQ' (Euc. I. 13), and the angle OSQ is the supplement of the angles SOQ and SQO (Euc. I. 32). Therefore



$$\begin{aligned}\text{the angle } HOQ' &= \text{the angle } OSQ\\ &= \text{the angle } OSQ'. \quad \text{[Prop. XVIII.}\end{aligned}$$

Ex. 1. Two tangents to a parabola, and the points of contact of one of them being given, prove that the locus of the focus is a circle.

The circle may be shown to pass through the given point of contact and the intersection of the tangents, and to touch one of them.

Ex. 2. If a parabola touch the sides of an equilateral triangle, the focal distance of any vertex of the triangle passes through the point of contact of the opposite side.

Ex. 3. Given the base AB and the vertical angle C of a triangle ACB, find the locus of the focus of a parabola touching CA, CB in A and B.

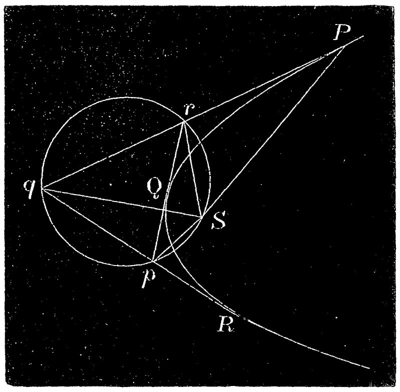
Ex. 4. E is the centre of the circle described about the triangle

OQQ'; prove that the circle described about the triangle *QEQ'* passes through the focus.

Ex. 5. A circle passing through the focus cuts the parabola in two points. Prove that the exterior angle between the tangents to the circle at those points is four times the complement of the exterior angle between the tangents to the parabola at the same points.

* PROPOSITION XX.

The circle circumscribing the triangle formed by any three tangents to a parabola passes through the focus.



Let the three tangents at the points P, Q, R form the triangle pqr.

Join SP, Sp, Sq, Sr.

The angle $Srp =$ the angle SPr, [Prop. XVIII.
and the angle $Sqp =$ the angle SPr; [Prop. XVIII.
therefore the angle $Srp =$ the angle Sqp.

Therefore the points p, q, r, S lie on a circle, or the circle round the triangle pqr passes through the focus.

Ex. 1. What is the locus of the focus of a parabola which touches three given straight lines?

Ex. 2. A parabola touches each of four straight lines given in position. Determine its focus.

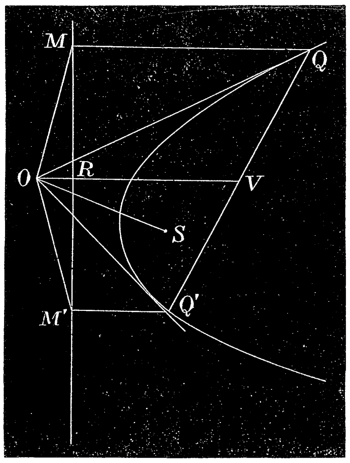
The four circles circumscribing the four triangles formed by the given straight lines, will intersect in the same point, namely, the focus required. Hence, the curve may be described. (See Prop. XXIII., Ex. 5.)

Ex. 3. If through p, q, r lines be drawn at right angles to Sp, Sq, Sr respectively, they will meet in a point.

Ex. 4. Prove that the orthocentre of the triangle pqr lies on the directrix. (Apply Prop. XII.)

* PROPOSITION XXI.

If through the point of intersection of two tangents to a parabola a straight line be drawn parallel to the axis, it will bisect the chord of contact.



Let OQ and OQ' be the two tangents, and let OV drawn parallel to the axis meet QQ' in V and the directrix in R. Draw QM and $Q'M'$ perpendicular to the directrix, and join OS, OM, OM'.

Then $OM = OS = OM'$, [Prop. XVII.
and OR, which is drawn at right angles to the base of the isosceles triangle OMM', bisects it.

Therefore $MR = M'R$.
But since MQ, RV, $M'Q'$ are parallel to one another,

$$QV : Q'V = MR : M'R;$$

therefore $QV = Q'V$,
or, QQ' is bisected in V.

Ex. 1. The tangents at the extremities of any chord of a parabola meet on the diameter bisecting that chord.

Ex. 2. The circle on any focal chord as diameter touches the directrix.

Ex. 3. The straight lines drawn through the extremities of a focal chord at right angles to the tangents at those points, meet on the diameter bisecting the chord.

Ex. 4. Given two tangents and their points of contact, find the focus and directrix.

Ex. 5. Given two points P, Q on a parabola, the tangent at one of the points P, and the direction of the axis, construct the curve.

If the tangent at P meets the diameter bisecting PQ in T, TQ is the tangent at Q. Hence the focus by Prop. XIV.

Ex. 6. If a line be drawn parallel to the chord of contact of two tangents, the parts intercepted on it between the curve and the tangents are equal.

Ex. 7. OP, OQ are two tangents to a parabola, and V is the middle point of PQ. Prove that $OP \cdot OQ = 2OS \cdot OV$.

On QO produced take $OQ' = OQ$; then apply Prop. XVIII. to show that the triangles POQ' and OSQ are similar.

Ex. 8. If from any point O a tangent OT and a chord OPQ be drawn, and if the diameter TR meet the chord in R, prove that $OP \cdot OQ = OR^2$. (Cf. *Tangent Properties*, Ex. 1, 2.)

Draw the tangent $KO'P'$ parallel to the chord, meeting RT in K, OT produced in O, and the curves in P'. Draw the diameter $O'H$ bisecting TP', so that $O'P' = KO'$. Then

$$OP \cdot OQ : OT^2 = O'P'^2 : O'T^2 = OK^2 : O'T^2 = OR^2 : OT^2.$$

Ex. 9. Given a chord PQ of a parabola in magnitude and position, and the point R in which the axis cuts the chord, the locus of the vertex is a circle.

If the tangent at the vertex meets PQ in O, $OP \cdot OQ = OR^2$.
$\therefore$ O is a fixed point; $OR = PR \cdot RQ/(PR - RQ)$.

Ex. 10. The tangents from an external point are divided by any third into segments having the same ratio.

In fig. Prop. XX., draw the diameters rr', QQ', qq', pp', meeting PR in r', Q', q', p'. Then

$$Pr : rq = rQ : Qp = qp : pR.$$

(Cf. Prop. XVIII., Ex. 11.)

Ex. 11. The tangent parallel to QQ' bisects OQ, OQ'.

Ex. 12. If E be the centre of the circle through O, Q, Q', OE subtends a right angle at S. (Apply Prop. XX., and Ex. 11.)

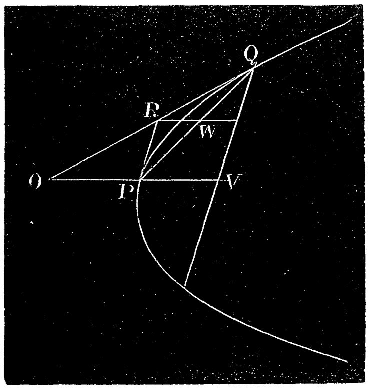
Ex. 13. If OQQ' be a right angle and QN the ordinate of Q, prove that

$$QQ' : OQ = QN : AN.$$

(Cf. Prop. XVI.)

*Proposition XXII.

If QV is the ordinate of a diameter PV of a parabola, and the tangent at Q meets VP produced in O, then OP shall be equal to PV.



Let the tangent at P meet OQ in R; through R draw the diameter RW, meeting PQ in W.

Then, since RP, RQ are a pair of tangents,

$$QW = PW. \qquad \text{[Prop. XXI.}$$

Also, RP is parallel to QV; [Prop. XI.

therefore $OP : PV = OR : RQ$

$= PW : WQ.$

But $PW = WQ;$

therefore $OP = PV.$

Ex. 1. Tangents at the extremities of all parallel chords meet on the same straight line. (Cf. Prop. XXI., Ex. 1.)

Ex. 2. Given a tangent and a point on the curve, find the locus of the foot of the ordinate of the point of contact of the tangent, with respect to the diameter through the given point. [A right line parallel to the tangent.]

Ex. 3. If $OV = QV$, O is on the directrix.

Ex. 4. If the diameter PV meets the directrix in O, and the chord drawn through the focus parallel to the tangent at P in V, prove that $VP = OP$.

Ex. 5. If OQ, OQ' be a pair of tangents to a parabola, and OQQ be a right angle, OQ will be bisected by the directrix.

Draw the diameter OPV and the tangent at P. (See Prop. XVI., Ex. 9.)

Ex. 6. If QV be an ordinate to the diameter PV, and pv meeting PQ in v be the diameter bisecting PQ, prove that $PV = 4pv$.

Ex. 7. PQ, PR are any two chords; they meet the diameters through R and Q in F and E. Show that EF is parallel to the tangent at P.

Ex. 8. If from the point of contact of a tangent a chord be drawn, and any line parallel to the axis be drawn meeting the tangent, curve, and chord, this line will be divided by them in the same ratio as it divides the chord.

Let the diameter RBV bisecting the chord QQ' in V meet the tangent at Q in R. Draw the line rbv parallel to the axis, cutting the curve and chord in b and v. Then

$Qv : vr = QV : VR$

$= QV : 2VB.$

But $QV^2 = 4BS \,.\, BV;$ (Prop. VII.)

$\therefore$ $QV : 2BV = 2SB : QV;$

$\therefore$ $Qv \,.\, QV = 2SB \,.\, vr.$

Also $Qv \,.\, Q'v = 4SB \,.\, vb;$ (Prop. VIII.)

$\therefore$ $QQ' : Q'v = rv : vb;$

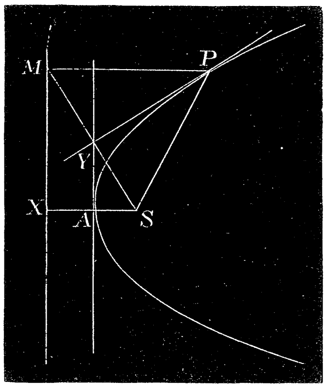
$\therefore$ $Qv : Q'v = rb : bv.$

This is a generalisation of Prop. XXII.

Ex. 9. Through a given point within a parabola, draw a chord which shall be divided in a given ratio at that point.

Proposition XXIII.

The locus of the foot of the perpendicular from the focus upon any tangent to a parabola is the tangent at the vertex.



Draw SY perpendicular to the tangent at P, meeting it in Y. It is required to show that Y lies on the tangent to the parabola at the vertex.

Draw PM perpendicular to the directrix, and join MY, AY.

Now, in the two triangles MPY, SPY, the sides MP, PY are equal to the sides SP, PY respectively, and the angle $MPY =$ the angle SPY. [Prop. XIV.

Therefore the angle $PYM =$ the angle PYS

$=$ one right angle; [Euc. I. 4.

therefore MY and YS are in the same straight line. [Euc. I. 14.

Now, since $SY = YM$,

and $SA = AX$,

AY is parallel to MX, [Euc. VI. 2.

and is, therefore, the tangent to the parabola at the vertex. [Prop. XIV., Cor.

Ex. 1. Show that $SY^2=AS.SP$. [The triangles SYP, SYA are similar. See Prop. XVIII., Ex. 1.]

Ex. 2. Show that SM is bisected at right angles by the tangent at P.

Ex. 3. If the tangent at P meet the axis in T, and PN be the ordinate of P, prove that $PT.TY=NT.TS$.

Ex. 4. If the vertex of a right angle, one leg of which always passes through a fixed point, moves along a fixed right line, the other leg will always touch a parabola.

The fixed point will be the focus, and the fixed right line the tangent at the vertex, whence the directrix is known.

Ex. 5. Given two tangents and the focus of a parabola, find the directrix.

The line joining the feet of the perpendiculars from the focus on the given tangents, is clearly the tangent at the vertex.

Ex. 6. Prove that straight lines perpendicular to the tangents of a parabola through the points where they meet a given fixed line parallel to the directrix, touch a confocal parabola.

Ex. 7. The focus and a tangent being given, the locus of the vertex is a circle.

Ex. 8. Given a tangent and the vertex, find the locus of the focus. [A parabola, of which A is the vertex and the axis the perpendicular through A on the tangent. Apply Prop. VII. Ex. 7.]

Ex. 9. The circle described on any focal distance as diameter, touches the tangent at the vertex.

Ex. 10. PSp is a focal chord; prove that the length of the common tangent of the circles described on Sp, SP as diameters, is

$$\sqrt{(AS.Pp)}.$$

Ex. 11. Prove that

(i.) $PY.PZ=PS^2$;

(ii.) $PY.YZ=AS.SP$.

Ex. 12. A circle is described on the latus rectum as diameter; PQ touches the parabola at P and the circle at Q; show that SP SQ are each inclined to the latus rectum at an angle of 30°.

Ex. 13. *Alternative Construction for Prop. XVII.*

Let O be the external point; on OS as diameter describe a circle; the lines joining O with the points of intersection of this circle with the tangent at the vertex, will be the required tangents.

Ex. 14. In the figure of Prop. VII., prove that $QD^2=4AS.BV$.

Let the tangent at B meet the axis in T, and the tangent at A in

Y. Then SYZ is a right angle, and the triangles QDV, YAT are similar (Prop. XI.)

$$\therefore \quad QD^2 : QV^2 = YA^2 : YT^2 = AS : TS = AS : BS.$$

But $$QD^2 = 4BS \,.\, BV. \quad \text{(Prop. VII.)}$$

$$\therefore \quad QV^2 = 4AS \,.\, BV.$$

Ex. 15. Given the focus and two tangents, construct the curve. [Ex. 5].

Ex. 16. Given the focus, axis and a tangent, construct the parabola.

Ex. 17. Given the focus, a point P on the parabola, and the length of the perpendicular from the focus on the tangent at P, construct the curve.

Ex. 18. Given the focus, a tangent, and the length of the latus rectum, construct the curve.

Ex. 19. If a parabola roll upon another equal parabola, the vertices originally coinciding, the focus of the one traces out the directrix of the other. [The line joining the foci in any position cuts at right angles the common tangent.]

PROPERTIES OF NORMALS.

Def. The straight line which is drawn through any point on a conic at right angles to the tangent at that point is called the *normal* at that point.

Def. The portion of the axis intercepted between the normal at any point of a conic and the ordinate of that point is called the *subnormal*.

Proposition XXIV.

The normal at any point of a parabola makes equal angles with the focal distance and the axis.

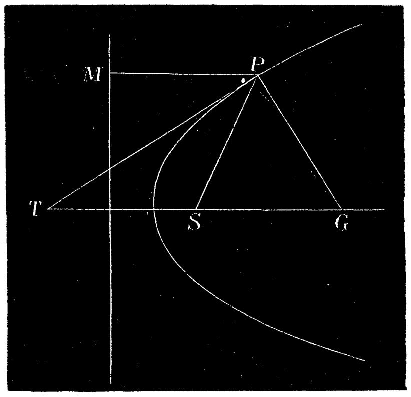
Let the normal PG and the tangent PT at any point P on the parabola meet the axis in G and T respectively. Join SP and draw PM perpendicular to the directrix.

Then the angle SPT = the angle TPM [Prop. XIV.

= the angle STP. [Euc. I. 29.

But the angle *TPG* being a right angle is equal to the sum of the angle *STP* and *SGP*. [Euc. I. 32.

Therefore the angle $SPG =$ the angle *SGP*.



Ex. 1. Prove that $ST = SP = SG$.

Ex. 2. The normal at any point bisects the interior angle tween the focal distance and the diameter through that point.

Ex. 3. The focus is equidistant from *PT* and the straight line through *G* parallel to *PT*.

Ex. 4. From the points where the normals to a parabola meet the axis, lines are drawn at right angles to the normals; show that these lines touch an equal confocal parabola.

Ex. 5. A chord *PQ* of a parabola is normal to the curve at *P*, and subtends a right angle at *S*; show that $SQ = 2SP$.

Ex. 6. Prove that *SM* and *PT* bisect each other at right angles.

Ex. 7. If the triangle *SPG* is equilateral, *TG* subtends a right angle at *M*.

Ex. 8. Prove that the points *S*, *P*, *M*, *Z* lie on a circle which touches *PG* at *P*.

Ex. 9. If in Ex. 8 the radius of the circle is equal to *MZ*, the triangle *SPG* is equilateral.

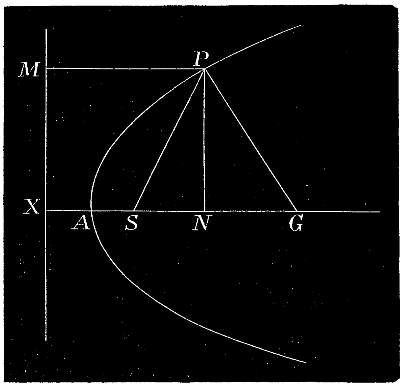
Ex. 10. *PSp* is a focal chord; *pG* is the normal at *p*; *GH* is perpendicular on the tangent at *P*. Prove that *H* lies on the latus rectum. (Cf. Prop. XIV., Ex. 10.)

Ex. 11. If *PF*, *PH* be drawn to the axis, making equal angles with the normal PG, prove that $SG^2 = SF \cdot SH$. [The triangles SPF, SHP are similar.]

Ex. 12. If SY, SZ be perpendicular to the tangent and normal at P respectively, prove that YZ is a diameter.

Proposition XXV.

The subnormal of any point of a parabola is equal to half the latus rectum.



Let the normal PG at P meet the axis in G. Draw PM, PN perpendicular to the directrix and axis respectively. Join SP.

Then, the angle SPG = the angle SGP. [Prop. XXIV.

Therefore $SG = SP = PM = NX$,

Therefore $NG = XS = 2AS = \frac{1}{2}$ latus rectum. [Prop. II.

The subnormal is therefore of constant length.

Ex. 1. If the triangle SPG is equilateral, SP is equal to the latus rectum.

Ex. 2. Show how to draw the normal at any given point without drawing the tangent.

Ex. 3. If the ordinate of a point Q bisect the subnormal of a point P, the ordinate of Q is equal to the normal at P. (Apply Prop. IV.)

Ex. 4. Prove that $PG^2 = 4AS \cdot SP$.

Ex. 5. If C be the middle point of SG, prove that

$$CX^2 - CP^2 = 4AS^2.$$

Ex. 6. If PL perpendicular to AP meets the axis in L, prove that

$$GL = 2AS.$$

Ex. 7. TP, TQ are tangents to a given circle at P and Q. Construct a parabola which shall touch TP in P and have TQ for axis.

Ex. 8. The locus of the foot of the perpendicular from the focus on the normal is a parabola.

[Apply Prop. IV. SG is the axis, the vertex is at S, the latus rectum $= AS$.]

Ex. 9. If GK be drawn perpendicular to SP, prove that

$$PK = 2AS.$$

Ex. 10. Pp is a chord perpendicular to the axis; the perpendicular from p on the tangent at P meets the diameter through P in R; prove that $RP = 4AS$, and find the locus of R.

[The triangles PNG, RPp are similar. The locus of R is an equal parabola, having its vertex A' on the opposite side of X, such that $AA' = 4AS$.]

Ex. 11. A circle described on a given chord of a parabola as diameter cuts the curve again in two points; if these points be joined, the portion of the axis intercepted by the two chords is equal to the latus rectum.

Show also that, if the given chord is fixed in direction, the length of the line joining the middle points of the chords is constant.

[Apply Prop. VIII. The middle points of the chords are equidistant from the axis.]

MISCELLANEOUS EXAMPLES ON THE PARABOLA.

1. Find the locus of the point of intersection of any tangent to a parabola, with the line drawn from the focus, making a constant angle with the tangent.

2. OQ, OQ' are tangents to a parabola; V is the middle point of QQ'; OV meets the directrix in K, and QQ' meets the axis in N. Prove that $OKNS$ is a parallelogram.

3. Inscribe in a given parabola a triangle having its sides parallel to those of a given triangle.

4. Inscribe a circle in the segment of a parabola cut off by a double ordinate.

5. PGQ is a normal chord of a parabola, meeting the axis in G. Prove that the distance of G from the vertex, the ordinates of P and Q, and the latus rectum are four proportionals.

6. If AR, SY are perpendiculars from the vertex and focus upon any tangent, prove that

$$SY^2 = SY . AR + SA^2.$$

7. Describe a parabola touching three given straight lines and having its focus on another given line.

8. OP, OQ are tangents to a parabola at the points P, Q. If $SP + SQ$ is constant, prove that the locus of O is a parabola, and find its latus rectum.

9. Through any point on a parabola two chords are drawn, equally inclined to the tangent there; show that their lengths are proportional to the portions of their diameters intercepted between them and the curve.

10. The focal chord PSp is bisected at right angles by a line which meets the axis in O; show that $Pp = 2 . SO$.

11. On a tangent are taken two points equidistant from the focus; prove that the other tangents drawn from these points will intersect on the axis.

12. The locus of the centre of the circle circumscribing the triangle formed by two fixed tangents and any third tangent is a right line.

13. A chord PQ is normal to the parabola at P, and subtends a right angle at the vertex; prove that $SQ = 3 . SP$.

14. Given the vertex, a tangent, and the latus rectum, construct the parabola.

15. P, Q are variable points on the sides AC, AB of a given triangle, such that $AP:PC=BQ:QA$. Prove that PQ touches a parabola.

16. Apply properties of the parabola to prove that—

(i.) In any triangle the feet of the three perpendiculars from any point of the circumscribing circle on the sides lie on the same straight line.

(ii.) If four intersecting straight lines be taken three together, so as to form four triangles, the orthocentres of these triangles lie on a right line.

17. Describe a parabola through four given points.

18. A parabola rolls on an equal parabola, the vertices originally coinciding. Prove that the tangent at the vertex of the rolling parabola always touches a fixed circle.

19. If two intersecting parabolas have a common focus, the angle between their axes is equal to that which their common tangent subtends at the focus.

20. AP, AQ, are two fixed straight lines, and B a fixed point. Circles described through A and B cut the fixed lines in P and Q. Prove that PQ always touches a parabola with its focus at B.

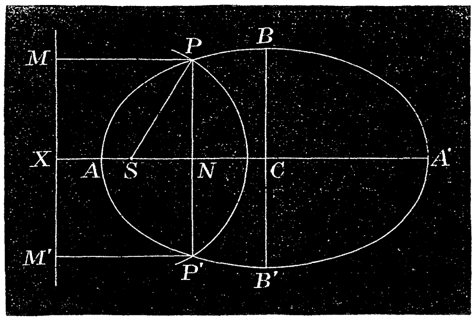
CHAPTER II.

THE ELLIPSE.

DESCRIPTION OF THE CURVE.

Proposition I.

Given the focus, directrix, and eccentricity of an ellipse to determine any number of points on it.



Let S be the focus, MXM' the directrix, and e the eccentricity.

Through S draw SX perpendicular to the directrix. Divide SX in A, so that

$$SA = eAX.$$

Also in XS' produced, take A' so that

$$SA' = eA'X.\text{*}$$

Then A and A' are points on the ellipse and are its vertices.

Take any point N on AA'; through N draw PNP' perpendicular to AA'; with centre S and radius equal to $e \,.\, XN$, describe a circle, cutting PNP' in P and P'. Then P and P' shall be points on the ellipse. Draw PM, $P'M'$ perpendicular to the directrix.

Then $$SP = e \,.\, XN \qquad \text{[Const.}$$
$$= e \,.\, PM,$$
and $$SP' = e \,.\, XN$$
$$= e \,.\, P'M'.$$

Therefore P and P' are points on the ellipse.

In like manner, by taking any other point on AA', any number of points on the curve may be determined.

Def. The length of the axis intercepted between the vertices (A and A') of the ellipse is called the *major axis*.

Def. The middle point (C) of the major axis is called the *centre* of the ellipse.

Def. The double ordinate (BCB') through the centre (c) is called the *minor axis* of the ellipse.

Ex. 1. The ellipse is symmetrical with respect to its axis.

Corresponding to any point N on the line AA' we get two points P and P', such that the chord PP' is bisected at right angles by the axis AA'.

Ex. 2. Any two right lines drawn from any point on the axis to the curve, on opposite sides of the axis and equally inclined to it, are equal, and conversely.

Ex. 3. If two equal and similar ellipses have a common centre, the points of intersection are at the extremities of central chords at right angles to each other.

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* Since e is less than unity it is clear that A will lie between X and S and A' without XS on the same side as S.

Ex. 4. Prove that the ellipse lies entirely between the lines drawn through A and A' at right angles to the axis.

In order that the circle may intersect PNP' the point N must be so situated that SN may not be greater than the radius of the circle SP, that is, eNX. It may easily be shown that this is the case only when N lies between A and A'.

Ex. 5. Show that as P moves from A to A', its focal distance (SP) increases from SA to SA'.

For $SP = e . NX$, and NX has AX and $A'X$ for its least and greatest values respectively.

Ex. 6. Hence prove that the ellipse is a closed curve.

Ex. 7. If a parabola and an ellipse have the same focus and directrix, the parabola lies entirely outside the ellipse.

Ex. 8. A chord QQ' of an ellipse meets the directrix in D. Prove that

$$SQ : QD = SQ' : Q'D.$$

Ex. 9. A straight line meets the ellipse at P and the directrix in D. From any point K in PD, KU is drawn parallel to DS to meet SP in U, and KI is drawn perpendicular to the directrix. Prove that $SU = e . KI$. (Cf. Prop. XVI., which is a particular case of this.)

Ex. 10. A point P lies within, on or without the ellipse, according as the ratio $SP : PM$ is less than, equal to, or greater than the eccentricity, PM being the perpendicular on the directrix.

Proposition II.

The ellipse is symmetrical with respect to the minor axis and has a second focus (S') and directrix.

Let S be the given focus and MX the given directrix.

Take any point M on the directrix, and through the vertices A and A' draw AH and $A'H'$ at right angles to AA', meeting the straight line through M and S at H and H' respectively. Describe a circle on HH' as diameter and through M draw MPP', parallel to AA', to meet the circle in P and P'. Then P and P' shall be points on the ellipse.

For $\qquad MH : HS = XA : AS = 1 : e,$

and $\qquad MH' : H'S = XA' : A'S = 1 : e.$

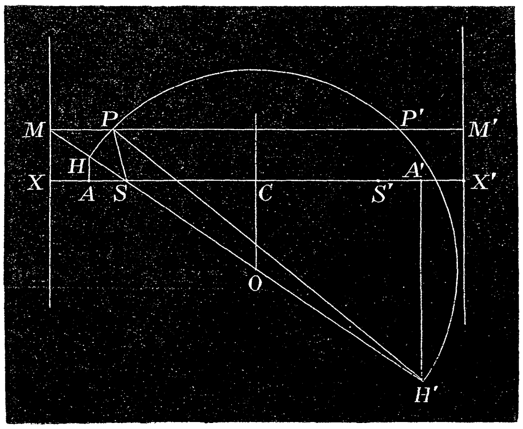
Therefore $MH : HS = MH' : H'S$,

and the angle HPH' is a right angle. [Euc. III. 31.

Therefore, PH bisects the angle SPM.

Therefore $$\begin{aligned} SP : PM &= SH : HM \\ &= AS : AX \\ &= e. \end{aligned}$$

Therefore, P is a point on the ellipse. Similarly, it may be shown that P' is a point on the ellipse.



Again, the straight line drawn through O, the centre of the circle, at right angles to AA' will bisect both AA' and PP' at right angles, and will therefore coincide with the minor axis in position.

The ellipse is therefore symmetrical with respect to the minor axis. [Def.

As the minor axis divides the curve into two parts such that each is the exact reflexion of the other, if $A'S'$ be measured off equal to AS and $A'X' = AX$, and $X'M'$ be drawn at right angles to $X'X$, the curve could be equally well described with S' as focus and $X'M'$ as directrix.

The ellipse therefore has a second focus (S') and a second directrix ($X'M'$).

Ex. Every chord drawn through the centre C is bisected at that point. (From the symmetry of the figure.)

From this property the point C is called the centre of the curve.

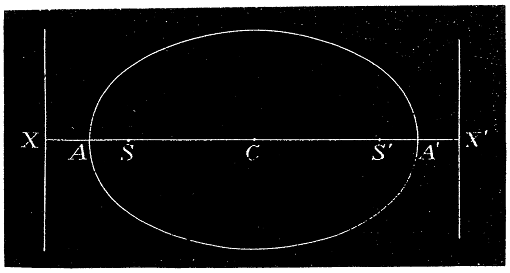
PROPERTIES OF CHORDS AND SEGMENTS OF CHORDS.

Proposition III.

In the ellipse $CA = e \,.\, CX$ ....(1)

$CS = e \,.\, CA$ ....(2)

$CS \,.\, CX = CA^2$ ....(3)



We have, from the definition,

$$SA = e \,.\, AX,$$

$$SA' = e \,.\, A'X = e \,.\, AX'.$$

Therefore, by addition,

$$AA' = e(AX + AX')$$

$$= eXX'.$$

Therefore $CA = e \,.\, CX$ ....(1)

By subtraction, $SS' = e(A'X - AX)$

$= e \,.\, AA'.$

Therefore $CS = e \,.\, CA. \qquad\qquad (2)$

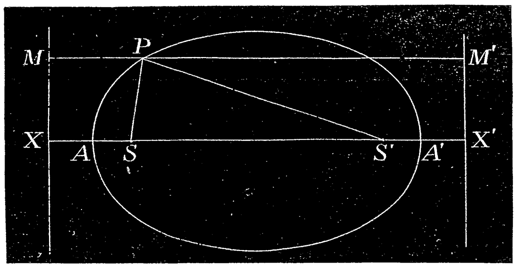
Therefore $CS \,.\, CX = CA^2. \qquad\qquad (3)$

Ex. Given the ellipse and one focus, find the centre and the eccentricity.

Describe a circle with S as centre, cutting the curve in P, P'. The axis bisects PP' at right angles.

Proposition IV.

The sum of the focal distances of any point on an ellipse is constant and equal to the major axis.



Let P be any point on the ellipse. Join PS, PS', and through P draw MPM' perpendicular to the directrices.

Then $SP = e \,.\, PM,$

$S'P = e \,.\, PM'.$

Therefore $SP + S'P = e \,.\, (PM + PM')$

$= e \,.\, MM'$

$= eXX'$

$= AA'.$ [Prop. III.

Ex. 1. Show how to construct the ellipse mechanically.

First Method.—Fasten the ends of a string to two drawing pins fixed at S and S' on a board, and trace a curve on the board with a pencil pressed against the string, so as to keep it always stretched. The curve traced out will be an ellipse, with foci at S and S', and major axis equal to the length of the string.

Second Method.—Suppose two equal thin circular discs A and B, attached to each other, to rotate in opposite directions round an axis through their common centre; and, suppose one end of a fine string (which is wrapped round the discs, and passing through small rings at C and D in the plane of the discs, is kept stretched by the point of a pencil at P) to be wound on to its disc, while the other is wound off. The curve traced by P will have the property

$$CP + DP = \text{constant},$$

and will, therefore, be an ellipse.

Ex. 2. The sum of the focal distances of any point is greater than, equal to, or less than the major axis, according as the point is without, upon, or within the ellipse, and conversely.

Ex. 3. The distance of either extremity of the minor axis from either focus is equal to the semi-axis-major.

Ex. 4. A circle is drawn entirely within another circle. Prove that the locus of a point equidistant from the circumferences of the two circles, is an ellipse. [The centres will be the foci.]

Ex. 5. Two ellipses have a common focus, and their major axes equal. Show that they cannot intersect in more than two points.

The common points may be shown to lie on the line bisecting at right angles the line joining the second foci.

Ex. 6. Prove that the external bisector of the angle SPS' cannot meet the ellipse again, and is, therefore, the tangent to the ellipse at P, according to Euclid's conception of a tangent. (Cf. Prop. XVII.)

Prove also that every other line through P will meet the curve again. [Apply Ex. 2.]

Ex. 7. The major axis is the longest chord that can be drawn in the ellipse.

Joining the foci with the extremities of any chord, it may be shown that twice the chord is less than the sum of the four focal distances, that is, less than twice the major axis.

Ex. 8. In what position of P is the angle SPS' greatest? [When P is at either extremity of the minor axis.]

Ex. 9. If r and R be the radii of the circles inscribed in and described about the triangle SPS', prove that Rr varies as $SP \,.\, S'P$.

Proposition V.

In the ellipse

$$CB^2 = CA^2 - CS^2 = SA \,.\, SA'.$$

Let B be an extremity of the minor axis. Join BS, BS'.

Then $SB + S'B = AA'$. [Prop. IV.

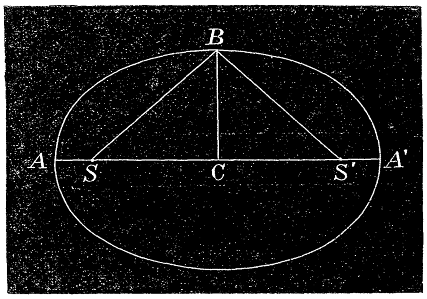
But $SB = S'B$.

Therefore $SB = CA$.

Therefore $CB^2 = SB^2 - CS^2$ [Euc. I. 47.

$= CA^2 - CS^2$

$= SA \,.\, S'A$. [Euc. II. 5.



Ex. 1. Prove that $e^2 = 1 - \frac{CB^2}{CA^2}$.

Ex. 2. Prove that $S'S^2 = A'A^2 - B'B^2$.

Ex. 3. If the angle SBS' be a right angle, show that
$$CA = \sqrt{2} \,.\, CB.$$

Ex. 4. A circle is described passing through B and touching the major axis in S; if SK be its diameter, prove that
$$SK \,.\, BC = AC^2.$$

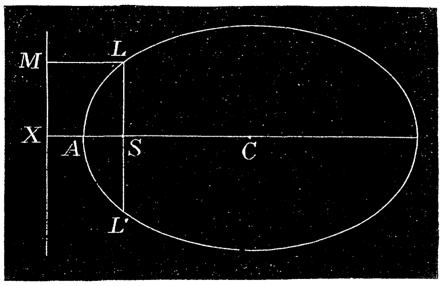
Ex. 5. Circles are described on the major and minor axes as diameters. PP' is a chord of the outer circle cutting the inner in Q, Q'. Prove that $PQ \,.\, P'Q = CS^2$.

Ex. 6. Given a focus S and a point P on an ellipse, and the lengths of the major and minor axes, find the centre.

On SP produced, take SK equal to the major axis; S' lies on the circle with centre P and radius PK. On SK as diameter describe a circle, and place in it KH equal to the minor axis; S' lies on the circle with centre S and radius SH.

* PROPOSITION VI.

The latus rectum of an ellipse is a third proportional to the major and minor axes ($SL = CB^2/CA$).



Let LSL' be the latus rectum. Draw LM perpendicular to the directrix.

Then $CS = eCA$, [Prop. III.

$SL = eLM$ [Def.

$= eSX$;

therefore $SL \,.\, CA = CS \,.\, SX$

$= CS(CX - CS)$

$= CS \,.\, CX - CS^2$

$= CA^2 - CS^2$ [Prop. III.

$= CB^2$. [Prop. V.

Ex. 1. Construct on the minor axis as base a rectangle which shall be to the triangle SLS' in the duplicate ratio of the major axis to the minor axis.

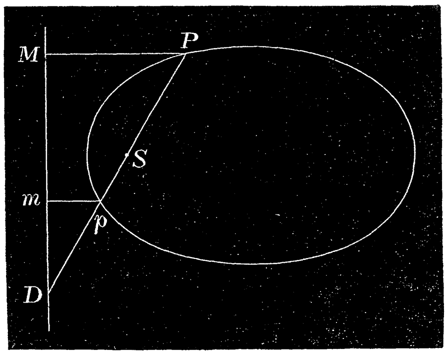
Draw BK parallel to LS', meeting the major axis in K; the other side of the rectangle $= \frac{1}{4}CK$.

Ex. 2. The extremities of the latera recta of all ellipses which have a common major axis, lie on two parabolas.

If LN be perpendicular to CB, $LN^2 = AC(AC - CN)$; hence, L lies on a parabola of which CB is the axis, and the vertex is at a distance from $C = CA$

*Proposition VII.

Any focal chord of an ellipse is divided harmonically by the focus and the directrix.



Produce the focal chord PSp to meet the directrix in D, and draw PM, pm perpendicular to the directrix.

Then $PD : pD = PM : pm$,

but $PS = e . PM$,

and $pS = e . pm$;

therefore $PD : pD = PS : pS$.

Hence Pp is divided harmonically in S and D.

Ex. 1. The semi-latus rectum is a harmonic mean between the segments of any focal chord.

Ex. 2. Focal chords are to one another as the rectangles contained by their segments.

Proposition VIII.

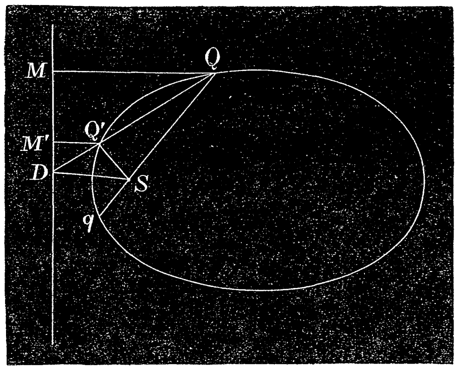
If any chord QQ' of an ellipse intersects the directrix in D, SD bisects the exterior angle between SQ and SQ'.

Draw QM, $Q'M'$ perpendiculars on the directrix, and produce QS to meet the ellipse in q.

Then, by similar triangles,

$$QD : Q'D = QM : Q'M'$$
$$= SQ : SQ';$$

therefore SD bisects the exterior angle $Q'Sq$. [Euc. VI. A.



Ex. 1. PSp is a focal chord. Prove that XP and Xp are equally inclined to the axis.

Ex. 2. Given the focus and three points on an ellipse, find the directrix and the axis.

Ex. 3. If P be any point on an ellipse, and PA, PA' when produced meet the directrix in E and F, show that EF subtends a right angle at the focus.

Ex. 4. If $A'S'$ be measured off along $A'A$ equal to AS, and $A'X'$ be measured off along AA' equal to AX, and if PA and PA' when produced meet the straight line through X' at right angles to the axis in E', F', show that $E'X' . F'X' = EX . FX$, and that $E'F'$ subtends a right angle at S'. (This is to be proved without assuming the existence of the second focus and directrix of the curve.)

Ex. 5. Hence, show that if PK be the perpendicular on $E'F'$, $S'P = e . PK$; and deduce the existence of a second focus and directrix corresponding to the vertex A'.

Ex. 6. If two fixed points Q, Q' on an ellipse be joined with a third variable point O on the curve, the segment qq' intercepted on either directrix by the chords QO and $Q'O$ produced, subtends a constant angle at the corresponding focus.

The angle qSq' may be proved to be equal to half of the angle QSQ'.

Ex. 7. *PSp* is a focal chord; *O* is any point on the curve; *PO*, *pO* produced meet the directrix in *D*, *d*. Prove that *Dd* subtends a right angle at the focus.

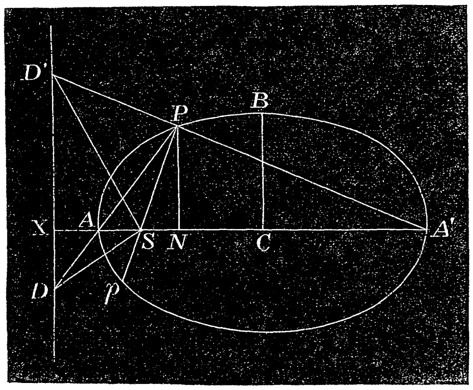
Ex. 8. Given the focus of an ellipse and two points on the curve, prove that the directrix will pass through a fixed point.

Ex. 9. A straight line which meets an ellipse will, in general, meet it in two points, and no straight line can meet it in more points than two.

The first part follows at once from the fact that the ellipse is a closed curve. (Prop. I., Ex. 6. Cf. also Ch. I., Prop. X., Ex. 8.) Then, if the line meets the curve in Q and Q', and the directrix in D, SQ and SQ' will be equally inclined to DS. Hence, if there be a third point of intersection Q'', SQ' and SQ'' will make the same angle with DS, which is impossible.

PROPOSITION IX.

The square of the ordinate of any point on an ellipse varies as the rectangle under the segments of the axis made by the ordinate ($PN^2 : AN \cdot A'N = CB^2 : CA^2$).



Let PN be the ordinate of any point P on the ellipse. Let PA and $A'P$ produced meet the directrix in D and D'. Join SD, SD', and SP, and produce PS to meet the curve in p.

Then, from the similar triangles PAN and DAX,

$$PN : AN = DX : AX.$$

Also, from the similar triangles $PA'N$ and $D'A'X$,

$$PN : A'N = D'X : A'X;$$

therefore $\quad PN^2 : AN . A'N = DX . D'X : AX . A'X.$

Again, SD and SD' bisect the angles pSX and PSX respectively; [Prop. VIII.

therefore the angle DSD' is a right angle, and

$$DX . D'X = SX^2; \qquad \text{[Euc. VI. 8.}$$

therefore $\quad PN^2 : AN . A'N = SX^2 : AX . A'X.$

But the ratio $SX^2 : AX . A'X$ is constant; therefore the ratio $PN^2 : AN . A'N$ has the same value for all positions of P.

In the particular case when P coincides with the extremity B of the minor axis, the ratio $PN^2 : AN . A'N$ becomes $CB^2 : CA^2$; therefore

$$PN^2 : AN . A'N = CB^2 : CA^2,$$

P being any point on the ellipse.

Ex. 1. Prove that $PN^2 : CA^2 - CN^2 = CB^2 : CA^2$.

Ex. 2. Prove that $\dfrac{CN^2}{CA^2} + \dfrac{PN^2}{CB^2} = 1$,

Ex. 3. Prove that $CP^2 = CB^2 + e^2 . CN^2$; and hence deduce that of all lines drawn from the centre to the curve CA is the greatest and CB the least. (See Prop. V., Ex. 1.)

Ex. 4. Show that PN increases as N moves from A to C.

Ex. 5. If PM be drawn perpendicular to the minor axis, deduce that $PM^2 : BM . B'M = CA^2 : CB^2$.

Ex. 6. P, Q are two points on an ellipse. AQ, $A'Q$ cut PN in L and M respectively. Prove that $PN^2 = LN . MN$.

Ex. 7. Deduce Prop. VI.

Ex. 8. If NQ be drawn parallel to AB, meeting the minor axis in Q, show that $PN^2 = BQ . B'Q$.

Ex. 9. If a point P moves such that $PN^2 : AN . A'N$ in a constant ratio, PN being the distance of P from the line joining two fixed points A, A', and N being between A and A', the locus of P is an ellipse of which AA' is an axis.

Ex. 10. The locus of the intersection of lines drawn through A, A' at right angles to AP, $A'P$, is an ellipse. [AA' will be the minor axis. See Ex. 5, 9.]

Ex. 11. The tangent at any point P of a circle meets the tangent at the extremity A of a fixed diameter AB in T. Find the locus of the point of intersection (Q) of AP and BT.

QM being perpendicular to AB, the triangles QMA, APB, and ATC are similar; so are the triangles QMB and TAB. Hence

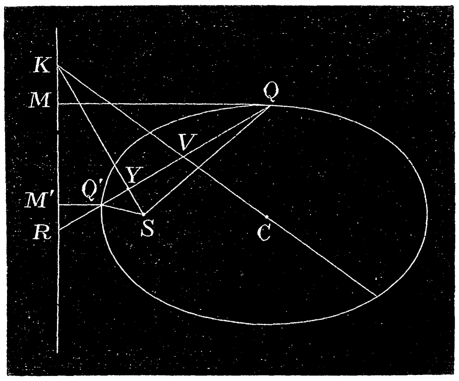
$$QM^2 : AM . BM = AC : AB.$$

Ex. 12. The ordinates of all points on an ellipse being produced in the same ratio, the locus of their extremities is another ellipse.

Ex. 13. P is any point on an ellipse; AQO is drawn parallel to CP meeting the curve in Q and CB produced in O. Prove that $AO . AQ = 2CB^2$.

PROPOSITION X.

The locus of the middle points of any system of parallel chords of an ellipse is a straight line passing through the centre.



Let QQ' be one of a system of parallel chords and V its middle point.

Draw QM, $Q'M'$ perpendicular to the directrix. Draw

SY perpendicular to QQ' and produce it to meet the directrix in K. Produce QQ' to meet the directrix in R. Join SQ, SQ'.

Then $$SQ : SQ' = QM \cdot Q'M' \\ = QR : Q'R.$$

Therefore $SQ^2 - SQ'^2 : QR^2 - Q'R^2 = SQ^2 : QR^2$.

But $$SQ^2 - SQ'^2 = QY^2 - Q'Y^2 \\ = (QY + Q'Y)(QY - Q'Y) \\ = 2QQ' . YV.$$

Similarly $QR^2 - Q'R^2 = 2QQ' . RV$,

Therefore $YV : RV = SQ^2 : QR^2$.

Now the ratio $SQ : QM$ is constant, also the ratio $QM : QR$ is constant, since QQ' is drawn in a fixed direction. Therefore $SQ : QR$ is a constant ratio.

Therefore also $YV : RV$ is a constant ratio for all chords of the system.

But as R always lies on a fixed straight line (the directrix) and Y on another fixed straight line (the focal perpendicular on the parallel chords) intersecting the former in K, V must also lie on a third fixed straight line passing through the same point K.

Also C, the centre of the ellipse, is evidently a point on this line, since the parallel chord through C is, from the symmetry of the figure, bisected at that point.

Hence, the diameter bisecting any system of parallel chords of an ellipse is a chord passing through its centre.

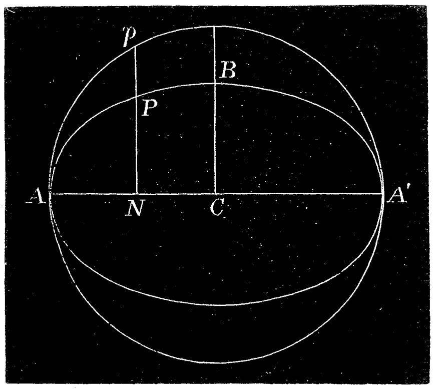
Ex. The diameter bisecting any system of parallel chords, meets the directrix on the focal perpendicular on the chords.

Note.—See Prop. XI., Ex. 10.

Def. The circle described on the major axis (AA') as diameter is called the *auxiliary circle*.

PROPOSITION XI.

Ordinates drawn from the same point on the axis to the ellipse and the auxiliary circle are in the ratio of the minor to the major axis.



Let ApA' be the auxiliary circle and let NPp be a common ordinate to the ellipse and the circle.

Then $PN^2 : AN \cdot A'N = CB^2 : CA^2$, [Prop. IX.

and $pN^2 = AN \cdot A'N$. [Euc. III. 3 & 35.

Therefore $PN^2 : pN^2 = CB^2 : CA^2$.

Therefore $PN : pN = CB : CA$.

Note.—By the *help* of this important property of the circle upon the major axis as diameter, many propositions concerning the ellipse may be easily proved, as will be shown hereafter. Hence the name *auxiliary* circle.

Def. The points P and p lying on a common ordinate pPN of the ellipse and its auxiliary circle are called *corresponding points.*

Ex. 1. A straight line cannot meet the ellipse in more than two points. (Cf. Prop. VIII., Ex. 9.)

Ex. 2. PM drawn perpendicular to BB' meets the circle on the minor axis as diameter in p'. Prove that

$$PM : p'M = CA : CB.$$

(See Prop. IX., Ex. 5.)

Ex. 3. PN, PM are perpendiculars on the axes, meeting the circles on the axes as diameters in p, p' respectively.

Prove that p and p' being properly selected, pp' passes through the centre.

Ex. 4. Through P, KPL is drawn making the same angle with the axes as pC, and cutting them in K and L. Show that KL is of constant length. ($KL = CA + CB$.)

Ex. 5. If the two extremities of a straight line move along two fixed straight lines at right angles to each other, any given point on the moving line describes an ellipse.

Let the fixed straight lines intersect in O, and let P be the given point on the moving line AB of which C is the middle point. Let QPN drawn at right angles to OB, meet OC, OB in Q and N respectively. Then, since $OQ = AP$, the locus of Q is a circle; also, as $PN : QN = PB : PA$, the locus of P is an ellipse.

Ex. 6. Given the semi-axes in magnitude and position, construct the curve mechanically.

Mark off on the straight edge of a slip of paper two lengths PA and PB in the same direction and equal to the semi-axes respectively. If the paper be now made to move so that A and B may always be on the lines representing the axes in position, P will trace out the ellipse. (See Ex. 5.)

Ex. 7. If a circle roll within another circle of double its radius, any point in the area of the rolling circle traces out an ellipse.

First Method.—Let C be the centre of the rolling circle, and O that of the other. If the given point P be on the radius CM, M will describe the diameter $A'OA$ of the outer circle. Draw RPN perpendicular to OA', meeting OC in R and OM in N. Then since $CR = CP$, the locus of R is a circle; and, as $PN : RN = PM : OR$, the locus of P is an ellipse.

Second Method.—The point M coincided with A' at the beginning of the motion; if in any position, the circles touch at Q,

$$\text{arc } MQ = \text{arc } A'Q, \qquad \text{angle } QCM = 2 \text{ angle } QOM,$$

$\therefore$ OCQ is always a straight line, so also is MCN, N being the intersection of the inner circle with that radius of the outer which is at right angles to OA. It is clear, therefore, that the motion of a point P in MN is exactly the same as that of a point in the moving rod in Ex. 5.

Ex. 8. From the centre of two concentric circles, a straight line is drawn to cut them in P and Q; through P and Q straight lines are drawn parallel to two given lines at right angles. Prove that the locus of their point of intersection is an ellipse, of which the outer circle is the auxiliary circle.

Ex. 9. NPp, $N'P'p'$ are ordinates of the ellipse and its auxiliary circle. Show that PP', pp' produced meet on the axis in the same point T.

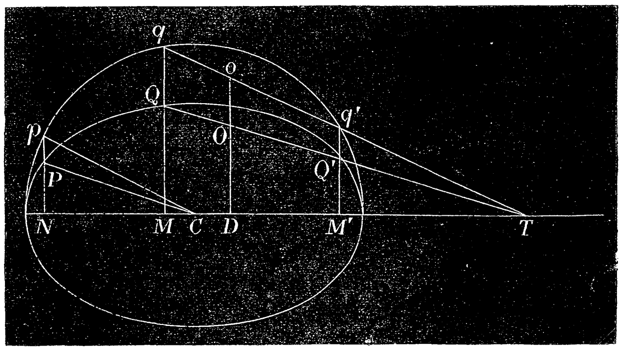
Ex. 10. Deduce from Ex. 9 a proof of Prop. X.

Let V, v be the middle points of PP', pp'. Vv produced bisects NN' at right angles in M. Now as long as PP' remains parallel to itself, pp' must remain parallel to itself, and, therefore, its middle point v lies on a fixed straight line, the diameter at right angles to pp'. V, therefore, lies on a fixed straight line through C, since

$$vM : VM = CB : CA.$$

* PROPOSITION XII.

If a system of chords of an ellipse be drawn through a fixed point the rectangles contained by their segments are as the squares of the parallel semi-diameters.



Let QOQ' be one of the system of chords drawn through the fixed point O and CP the semi-diameter parallel to QQ'. Then $QO \cdot OQ' : CP^2$ shall be a constant ratio.

Describe the auxiliary circle, and let p, q, q' be the corresponding points to P, Q, Q'. Join Cp and qq' and draw through O a line perpendicular to the major axis, meeting it in D and qq' in o.

Then, since $QM : qM = Q'M' : q'M'$

$= CB : CA$, [Prop. XI.

the straight lines QQ' and qq' produced meet the axis produced in the same point T.

Again, the triangles PNC and QMT being similar

$$NC : MT = PN : QM$$

$= pN : qM$. [Prop. XI.

Therefore the triangles pNC and qMT are similar. [Euc. VI. 6.

Therefore pC is parallel to qT.

Therefore the triangles pPC and qQT are also similar.

Now $QO : qo = QT : qT$,

also $OQ' : oq' = QT : qT$.

Therefore $QO \,.\, OQ' : qo \,.\, oq' = QT^2 : qT^2$

$= CP^2 : Cp^2$,

or $QO \,.\, OQ' : CP^2 = qo \,.\, oq' : CP^2$.

Now, since $OD : oD = CB : CA$,

and the point O is fixed, the point o is also fixed; hence $qo \,.\, oq'$ is constant. [Euc. III. 35.

Also $Cp = CA =$ constant.

Therefore $QO \,.\, OQ' : CP^2$

is a constant ratio.

Ex. 1. The ratio of the rectangles under the segments of any two intersecting chords of an ellipse, is equal to that of the rectangles under the segments of any other two chords parallel to the former, each to each.

Ex. 2. If two chords of an ellipse intersect, the rectangles under their segments are as the parallel focal chords. (Apply Prop. VII., Ex. 2.)

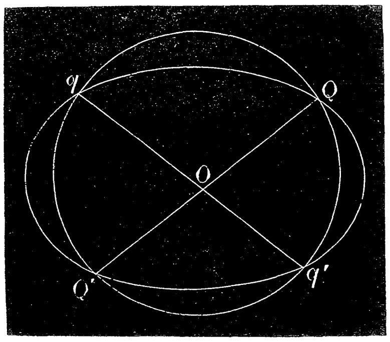
Ex. 3. Ordinates to any diameter at equal distances from the centre are equal.

Ex. 4. QCq is the central chord parallel to the focal chord PSp. Prove that

$$SP \,.\, Sp : CQ \,.\, Cq = CB^2 : CA^2.$$

* Proposition XIII.

If a circle intersect an ellipse in four points their common chords will be equally inclined, two and two, to the axis.



Let Q, Q', q, q', be the four points of intersection.

Join QQ', qq', intersecting in O.

Then $$QO \,.\, OQ' = qO \,.\, Oq', \qquad \text{[Euc. III. 35.}$$

Therefore the semi-diameters parallel to QQ' and qq' respectively, are equal to each other, [Prop. XII. and they are, therefore, equally inclined to the axis from the symmetry of the figure. (See also Prop. I., Ex. 2.) Therefore, the chords QQ' and qq' are equally inclined to the axis.

In like manner it may be shown that the chords Qq and $Q'q'$ as well as the chords Qq' and qQ' are equally inclined to the axis.

Ex. 1. If two chords, not parallel, be equally inclined to the axis of an ellipse, their extremities lie on a circle.

Ex. 2. If P be a fixed point on an ellipse and QQ' any ordinate to CP, show that the circle QPQ' will intersect the curve in another fixed point.

PROPERTIES OF TANGENTS.

It has been already observed in Chapter I. that, generally, from a chord property of a conic a corresponding tangent property may be deduced. The student should work out the following exercises as illustrating the method in the case of the ellipse.

* Deduce from Prop. XII.:—

Ex. 1. The tangents to an ellipse from an external point are proportional to the parallel semi-diameters.

Ex. 2. If the tangents at three points P, Q, R on an ellipse, intersect in r, q, p, show that

$$Pr . pQ . qR = Pq . rQ . pR.$$

Ex. 3. If two parallel tangents OP, $O'P'$ be met by any third tangent OQO', then $OP : O'P' = OQ . O'Q$.

Ex. 4. If from any point without an ellipse a secant and also a tangent be drawn, the rectangle under the whole secant and the external segment is to the square of the tangent as the squares of the parallel semi-diameters.

Ex. 5. If two tangents be drawn to an ellipse, any line drawn parallel to either will be cut in geometric progression by the other tangent, the curve and the chord of contact.

Ex. 6. Any two intersecting tangents to an ellipse are to one another in the sub-duplicate ratio of the parallel focal chords.

Ex. 7. If two parallel tangents AQ and OR be cut by any third tangent APO, and RP meets QA in B, show that $AQ = AB$.

*Deduce from Prop. XIII.:—

Ex. 1. PQ, PQ' are chords of an ellipse equally inclined to the axis. Prove that the circle PQQ' touches the ellipse at P.

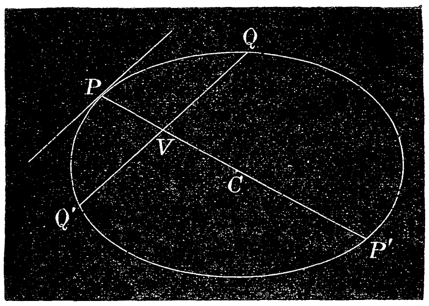
Ex. 2. PP' is a chord of an ellipse parallel to the major axis; $P'Q$, $P'Q'$ are chords equally inclined to that axis. Show that QQ' is parallel to the tangent at P.

Ex. 3. If a circle touch an ellipse at the points P and Q, prove that PQ is parallel to one of the axes.

See also Props. XIV. and XV.

PROPOSITION XIV.

The tangent to an ellipse at either end of a diameter is parallel to the system of chords bisected by the diameter.



Let $PVCP'$ be the diameter bisecting a system of chords parallel to QQ'. Let QQ' be made to move parallel to itself so that Q may coincide with V. Since QV is always equal to $Q'V$, [Prop. X.
it is clear that Q' will also coincide with V, and the chord in this its limiting position will be the tangent to the ellipse at P.

Ex. 1. The tangent at the vertex is at right angles to the major axis. [From symmetry, the chords at right angles to the major axis are bisected by it.]

Ex. 2. The line joining the points of contact of two parallel tangents is a diameter.

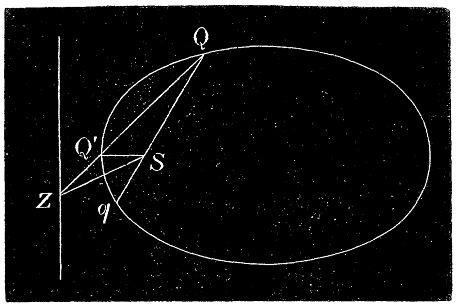
Ex. 3. Any tangent is cut harmonically by two parallel tangents and the diameter passing through their points of contact. (See note on Tangent Properties, I., Ex. 3.)

Ex. 4. An ellipse is described about the triangle ABC, having its centre at the point of intersection O of the medians. OA, OB, OC produced meet the ellipse in α, β, γ. Prove that the tangents at α, β, γ form a triangle similar to ABC and four times as large.

Proposition XV.

The portion of the tangent to an ellipse at any point intercepted between that point and the directrix subtends a right angle at the focus, and conversely.

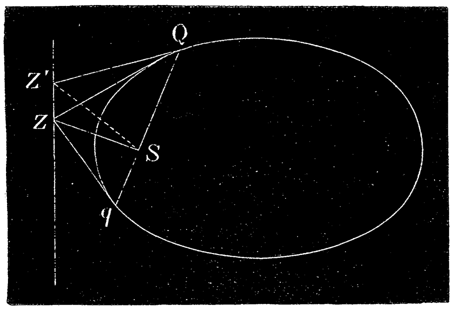
Also the tangents at the ends of a focal chord intersect on the directrix.



First.—Let any chord QQ' of the ellipse intersect the directrix in Z.

Then SZ bisects the exterior angle $Q'Sq$. [Prop. VIII. Now, let the chord QQ' be made to turn about Q until the point Q' moves up to and coincides with Q, so that the chord becomes the tangent to the ellipse at Q. In this limiting position of the chord QQ', since Q and Q' coincide, the angle QSQ' vanishes and therefore the angle $Q'Sq$ becomes equal to two right angles. But, since SZ always bisects the angle $Q'Sq$, in this case the angle QSZ is a right angle.

Again, let QZ subtend a right angle at S; then it shall be the tangent to the ellipse at Q. For, if not, and if possible, let QZ' be the tangent at Q; then the angle QSZ' is a right angle, which is impossible. Therefore QZ is the tangent at Q.



Secondly.—Let QSq be a focal chord and QZ the tangent at Q. Join ZS, Zq.

Then the angle QSZ being a right angle, the angle ZSq is also a right angle, and therefore qZ is the tangent to the ellipse at q. Therefore the tangents at Q and q intersect on the directrix.

Ex. 1. Tangents at the extremities of the latus rectum intersect in X.

Ex. 2. If through any point P of an ellipse, an ordinate QPN be drawn, meeting the tangent at L in Q, prove that $QN = SP$.

Ex. 3. To draw the tangent at a given point P of an ellipse.

Ex. 4. By drawing the tangent at B, prove that $CS \,.\, CX = CA^2$.

Ex. 5. If ZQ meets the other directrix in Z', $Z'P$ subtends a right angle at S'.

Ex. 6. If QZ intersect the latus rectum in D, prove that

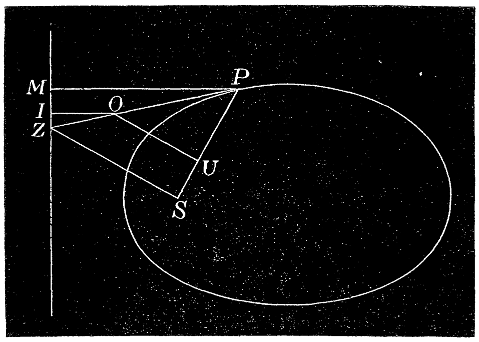
$$SD = e \,.\, SZ.$$

PROPOSITION XVI.

If from a point O on the tangent at any point P of an ellipse perpendiculars OU and OI be drawn to SP and the directrix respectively, then

$$SU = e \, . \, OI,$$

and conversely.



Join SZ and draw PM perpendicular to the directrix.

Because ZSP is a right angle, [Prop. XV. ZS is parallel to OU.

Therefore, by similar triangles,

$$SU : SP = ZO : ZP$$
$$= OI : PM.$$

But $$SP = e \, . \, PM;$$

therefore $$SU = e \, . \, OI.$$

Again, for the converse proposition, if a line OP meets the ellipse at P, and the same construction is made as before, we have

$$SU = e \, . \, OI,$$

and $$SP = e \, . \, PM;$$

therefore $$SU : SP = OI : PM$$
$$= ZO : ZP.$$

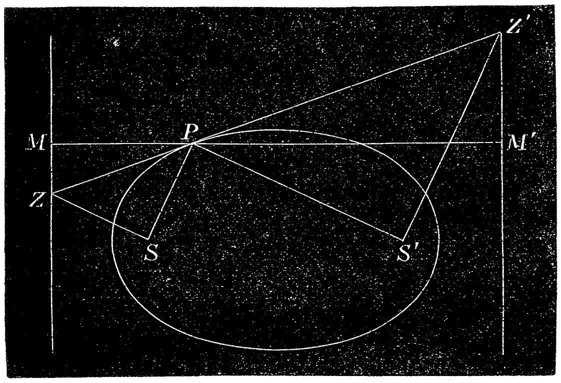
Therefore OU is parallel to ZS, [Euc. VI. 2.
and the angle PSZ is a right angle.

OP is, therefore, the tangent at P. [Prop. XV.

Note.—See Chap. I., Prop. XIII., also Prop. I., Ex. 9.

Proposition XVII.

The tangent at any point of an ellipse makes equal angles with the focal distances of the point.



Let the tangent at P meet the directrices in Z and Z'.

Draw MPM' perpendicular to the directrices, meeting them in M and M' respectively. Join SP, SZ, $S'P$, and $S'Z'$.

Then, in the two triangles PSZ and $PS'Z'$, the angles PSZ and $PS'Z'$ are equal, being right angles, [Prop. XV.
and

$$SP : S'P = PM : PM'$$
$$= PZ : PZ',$$

and the angles PZS and $PZ'S'$ are both acute angles.

Therefore the triangles are similar; [Euc. VI. 7.
therefore the angle $SPZ =$ the angle $S'PZ'$.

Ex. 1. If a line drawn through P bisect the exterior angle between SP and $S'P$, it will be the tangent at P.

Ex. 2. The tangent at the vertex is at right angles to the major axis.

Ex. 3. The perpendiculars from Z and Z' on SP intercept a length equal to AA'.

Ex. 4. The tangent at any point makes a greater angle with the focal distance than with the perpendicular on the directrix.

Ex. 5. If SY, $S'Y'$ be the perpendiculars upon the tangent at P, and PN be the ordinate of P, prove that PN bisects the angle YNY'.

Ex. 6. If SY, the perpendicular on the tangent at P, meet $S'P$ produced in s, prove that

(i) $sY = SY$, (ii) $SP = Ps$, (iii) $S's = AA'$.

On account of property (i), s is called the *image* of the focus in the tangent.

Ex. 7. Prove that the locus of the image of the focus in the tangent is a circle.

The circle, of which the centre is a focus and the radius equal to the major axis, is sometimes, though not quite properly, called the *Director Circle*, by way of analogy to the *directrix* of the parabola, which is, in the case of that curve, the locus of the image of the focus in the tangent. (See Chap. I., Prop. XIV., Ex. 7.)

Ex. 8. Given a focus and the length of the major axis, describe an ellipse touching a given straight line and passing through a given point. (Apply Prop. IV.; Newton, Book I., Prop. XVIII.)

Ex. 9. Given a focus and the length of the major axis, describe an ellipse touching two given straight lines. (Apply Prop. IV., cf. Prop. XXIII., Ex. 4; Newton, Book I., Prop. XVIII.)

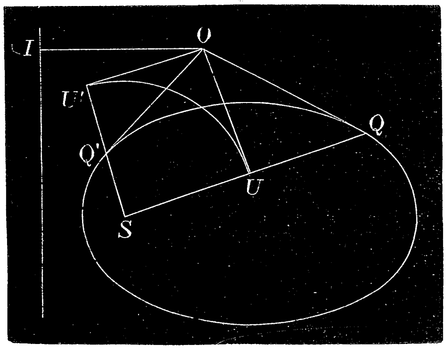
Ex. 10. If a circle be described through the foci of an ellipse, a straight line drawn from its intersection with the minor axis to its intersection with the ellipse, will touch the ellipse.

PROPOSITION XVIII.

To draw two tangents to an ellipse from an external point.

Let O be the external point. Draw OI perpendicular to the directrix, and with centre S and radius equal to

$e.OI$, describe a circle. Draw OU, OU' tangents to this circle, and let SU, SU' meet the ellipse in Q, Q'. Join OQ, OQ'. Then OQ, OQ' shall be the tangents required.



For OU is at right angles to SQ, [Euc. III. 18.
and $SU = e.OI$.
Therefore OQ is the tangent to the ellipse at Q.
[Prop. XVI.

Similarly OQ' is the tangent at Q'.

Ex. 1. *Alternative Construction.*—With centre O and radius OS describe a circle; with centre S' and radius equal to the major axis describe another circle intersecting the former in M and M'. Join $S'M$ and $S'M'$, meeting the ellipse in Q and Q'; OQ, OQ' are the tangents required. [The angle OQM = the angle OQS. Then apply Prop. XVII., Ex. 1. It may be shown that the construction given in Chap. I., Prop. XVI., is immediately deducible from this.]

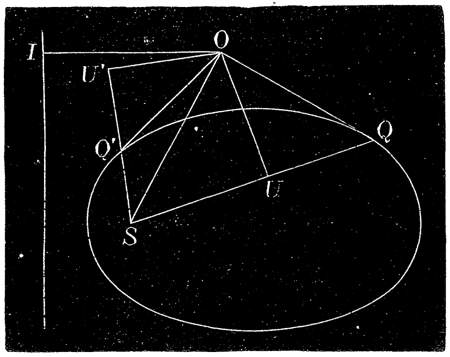
Ex. 2. Show that only two tangents can be drawn to an ellipse from an external point. (See Note to Chap. I., Prop. XVI.)

Proposition XIX.

The two tangents which can be drawn to an ellipse from an external point subtend equal angles at the focus.

Let OQ, OQ' be the two tangents from O.

Join SO, SQ, SQ', and draw OI, OU, OU' perpendiculars upon the directrix, SQ, SQ' respectively.



Then $SU = e . OI = SU'$. [Prop. XVI.

Therefore $OU = OU'$. [Euc. I. 47.

Therefore the angles OSU and OSU' are equal, [Euc. I. 8.

and they are the angles which the tangents subtend at the focus S.

Ex. 1. QQ' produced meets the directrix in Z. Prove that OZ subtends a right angle at S. [Prop. XV. is a particular case of this.]

Ex. 2. If P be any point on an ellipse, the centre of the circle touching the major axis, SP, and $S'P$ produced lies on the tangent at the vertex.

Ex. 3. The two foci and the intersections of any tangent with the tangents at the vertices, are concyclic points.

Ex. 4. A variable tangent meets a fixed tangent in T. Find the locus of the intersection with the variable tangent of the straight line through S at right angles to ST.

[The locus is the tangent at the other extremity of the focal chord through the point of contact of the fixed tangent.]

Ex. 5. The tangents at the ends of a focal chord meet the tangents at the vertex in T_1 and T_2. Prove that $AT_1 . AT_2$ is constant. $(=AS^2.)$

Ex. 6. The angle subtended at either focus by the segment intercepted on a variable tangent by two fixed tangents is constant.

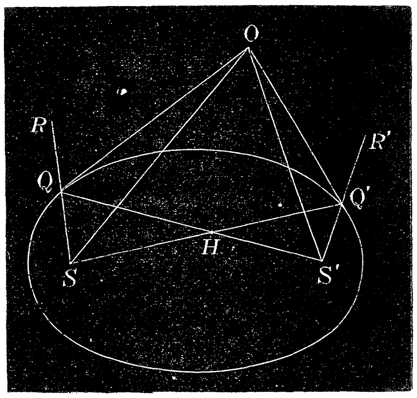
Ex. 7. If OS intersect QQ' in R and RK be drawn perpendicular to the directrix, prove that QK, $Q'K$ are equally inclined to the axis.

Ex. 8. An ellipse is inscribed in a triangle; if one focus moves along the arc of a circle passing through two of the angular points of the triangle, find the locus of the other focus. [An arc of a circle through the same angular points.]

Ex. 9. If a quadrilateral circumscribes an ellipse, the angles subtended by opposite sides at one of the foci are together equal to two right angles.

* PROPOSITION XX.

The two tangents drawn to an ellipse from an external point are equally inclined to the focal distances of that point.



Let OQ, OQ' be the two tangents from O.

Join SQ, SO, SQ', $S'Q$, $S'O$, $S'Q'$, and produce SQ to R. Let H be the point of intersection of SQ' and $S'Q$.

Then

the angle SOQ = the angle OQR − the angle OSQ [Euc. I. 32.

= half the angle $S'QR$ − half the angle QSQ' [Props. XVII. and XIX.

= half the angle SHQ.

Similarly,

the angle $S'OQ'$ = half the angle $S'HQ'$.

Therefore,

the angle SOQ = the angle $S'OQ'$. [Euc. I. 15.

Ex. 1. Given two tangents to an ellipse and one focus, show that the locus of the centre is a right line.

Ex. 2. On OQ, OQ' take OK, OK' equal to OS, OS' respectively. Prove that KK' is equal to the major axis. [If SQ produced to E be equal to the major axis, the triangles SOE and KOK' are equal.]

Ex. 3. The straight line joining the feet of the perpendiculars from a focus on two tangents is at right angles to the line joining the intersection of the tangents with the other focus.

Ex. 4. The exterior angle between any two tangents is half the sum of the angles which the chord of contact subtends at the foci. [Cf. Chap. I., Prop. XIX.]

Ex. 5. The angle between the tangents at the extremities of a focal chord is half the supplement of the angle which the chord subtends at the other focus.

Ex. 6. Prove that

$$\angle SOS' + \angle S'QO + \angle SQ'O = 2 \text{ right angles}.$$

Ex. 7. If from any point on an ellipse tangents are drawn to a confocal ellipse, these tangents are equally inclined to the tangent at that point.

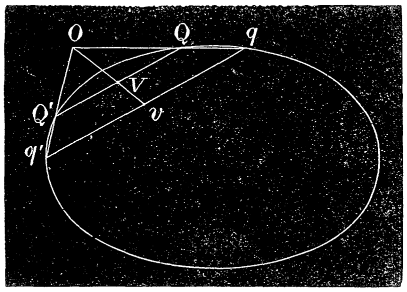
Def. Ellipses which have the same foci are called *confocal* ellipses.

Ex. 8. If a perfectly elastic billiard ball lies on an elliptic billiard table, and is projected in any direction along the table, show that the lines in which it moves after each successive impact touch a confocal conic.

Ex. 9. Normals at the extremities of a focal chord intersect in O, and the corresponding tangents meet in T. Prove that OT passes through the other focus.

PROPOSITION XXI.

The tangents at the extremities of any chord of an ellipse intersect on the diameter which bisects the chord.



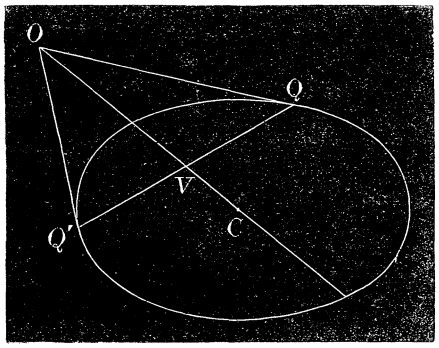
Let QQ' be the chord, and qq' any other chord parallel to it.

Let qQ and $q'Q'$ produced meet in O. Bisect QQ' in V and let OV meet qq' in v.

Then $$QV : qv = OV : Ov$$ $$= Q'V : q'v.$$

But $$QV = Q'V.$$

Therefore $$qv = q'v.$$



Therefore OVv is the diameter bisecting the system of chords parallel to QQ'. [Prop. X.

If now the chord qq' be made to move parallel to itself until it coincides with QQ', qQO and $q'Q'O$ will become the tangents to the curve at Q and Q' respectively, and they thus meet on the diameter bisecting QQ'.

Ex. 1. The diameter of an ellipse through an external point bisects the chord of contact of the tangents from that point.

Ex. 2. Given a diameter of an ellipse, to draw the system of chords bisected by it.

Ex. 3. The tangent at any point P of an ellipse meets the tangent at A in Y. Prove that CY is parallel to $A'P$.

Ex. 4. If $OPCP'$ be a diameter through O, OQ a tangent from O, and QV be drawn parallel to the tangent at P, then

$$OP \cdot OP' = OC \cdot OV.$$

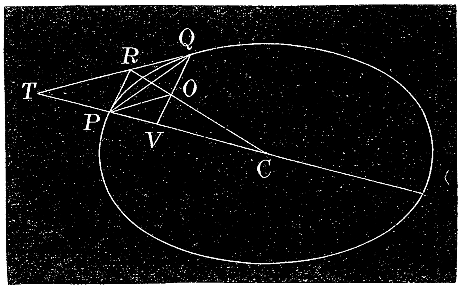
Hence show that $OP : OP' = PV : P'V$. [This shows that PP' is divided harmonically in V and O.]

Ex. 5. If any line drawn parallel to the chord of contact of two tangents to an ellipse meets the curve, the segments intercepted between the curve and the tangents are equal.

Proposition XXII.

If the tangent at any point Q of an ellipse meets any diameter CP produced in T, and if QV be the ordinate to that diameter,

$$CV \cdot CT = CP^2.$$



Draw the tangent PR at P, meeting QT in R, and draw PO parallel to QT meeting QV in O.

Then since $POQR$ is a parallelogram, RO bisects PQ, and therefore passes through the centre C.

[Prop. XIV. and XXI.

By similar triangles

$$CV : CP = CO : CR = CP : CT.$$

Therefore $$CV \,.\, CT = CP^2.$$

Note.—When the diameter coincides with the major axis, the result is stated thus :—

If the tangent at Q *meets the major axis produced in* T, *and* QN *be the perpendicular on the major axis,*

$$CN \,.\, CT = CA^2.$$

When the diameter coincides with the minor axis, the result is stated thus :—

If the tangent at Q *meets the minor axis produced in* t, *and* Qn *be the perpendicular on the minor axis,*

$$Cn \,.\, Ct = CB^2.$$

These two particular cases are important, and should be carefully noted by the student.

Ex. 1. VH drawn parallel to PQ meets CQ in R. Prove that PH is parallel to the tangent at Q.

Ex. 2. If a series of ellipses have the same major axis, the tangents at the extremities of their latera recta meet at the same point on the minor axis.

Ex. 3. If PT be a tangent to an ellipse meeting the axis in T, and AP, $A'P$ be produced to meet the perpendicular to the major axis through T in Q and Q', then $QT = Q'T$. [If PN be the ordinate of P, the relation $CT : CA = CA : CN$ gives $A'T : A'N = AT : AN$.]

Ex. 4. If PN be perpendicular to the major axis, and the tangent at P meet the major axis produced in T, any circle through N and T cuts the auxiliary circle at right angles. [If E be the centre of the circle, show that $EN^2 + CA^2 = EC^2$.]

Ex. 5. The locus of the middle points of all focal chords of an ellipse is a similar ellipse.

Let O be the middle point of a focal chord PSp, and let the tangent at Q where CO produced meets the curve, meet the major axis in T. If OM and QN be the ordinates to the major axis, it readily follows that

$$\frac{OM^2}{CM \, . \, SM} = \frac{QN^2}{CN \, . \, TN} = \frac{QM^2}{AN \, . \, A'N}.$$

Then apply Prop. IX., Ex. 9.

Ex. 6. If CY, AZ be the perpendiculars from the centre and an extremity of the major axis on the tangent at any point P, show that
$$CA \, . \, AZ = CY \, . \, AN.$$

Ex. 7. If a variable tangent to an ellipse meet two fixed parallel tangents, it will intercept segments on them whose rectangle is constant.

Let the tangent at Q meet the two parallel tangents PR and pr in R and r. Pp is a diameter (Prop. XIV., Ex. 2). Let CD be the semi-diameter parallel to PR meeting Rr in t. Let QV and Qv be ordinates to CP, CD; and let rR, pP meet in T. Then apply the proposition with respect to the diameters CD, CP.

Ex. 8. In Ex. 7 prove that the rectangle under the segments of the variable tangent is equal to the square of the semi-diameter drawn parallel to it. (See Note on *Tangent-Properties* Ex. 1, 2. Newton, Book I., Lemma XXIV.)

Ex. 9. If P is any point on the ellipse, find the locus of the centre of the circle inscribed in the triangle SPS'. [An ellipse. If ON be the perpendicular from the centre O on AA', it may be shown that
$$ON^2 : NS \, . \, NS' = SA^2 : CB^2.$$
Then apply Prop. IX., Ex. 9.]

Ex. 10. CD, CP are two semi-diameters of an ellipse. Tangents at D and P meet CP and CD in K and T respectively. Prove that the triangles CDK and CPT are equal in area.

Proposition XXIII.

The locus of the foot of the perpendicular drawn from either focus upon any tangent to an ellipse is the auxiliary circle; and the rectangle under the focal perpendiculars on the tangent is equal to the square of the semi-axis minor. ($SY \, . \, S'Y' = CB^2$.)

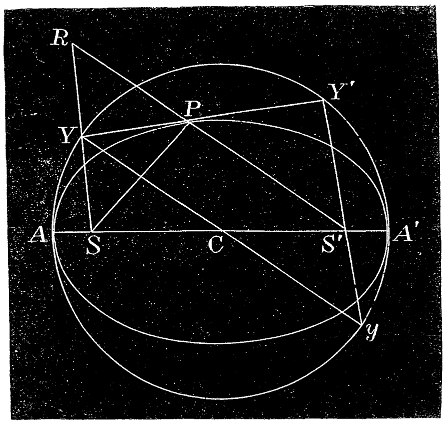
Let SY, $S'Y'$ be the focal perpendiculars upon the tangent at any point P.

Join SP and $S'P$. Produce $S'P$ to meet SY in R. Join CY.

Then in the triangles SPY, RPY,

the angle SPY = the angle $S'PY'$ [Prop. XVII.

= the angle RPY, [Euc. I. 15.



and the angles SYP, RYP are equal, each being a right angle, and YP is common,

therefore $SP = PR$.

and $SY = YR$. [Euc. I. 26.

Also $SC = CS'$,

therefore CY is parallel to $S'R$, [Euc. VI. 2.

therefore $CY = \frac{1}{2} . S'R$ [Euc. VI. 4.

$= \frac{1}{2}(S'P + PR)$

$= \frac{1}{2}(S'P + SP)$

$= \frac{1}{2}AA'$ [Prop. IV.

$= CA$.

Therefore the locus of Y is the auxiliary circle.

Similarly it may be shown that the locus of Y' is the same circle.

Again, produce YC and $Y'S'$ to meet in y, then y will be on the auxiliary circle.

For, since $CS = CS'$ and SY is parallel to $S'y$ the triangles SCY and $S'Cy$ are equal. [Euc. I. 26.
Therefore $Cy = CY = CA$, showing that y is on the auxiliary circle.

Also $$SY = S'y$$
Therefore $$SY . S'Y' = S'y . S'Y'$$
$$= S'A' . S'A \qquad \text{[Euc. III. 35.}$$
$$= SA . SA'$$
$$= CB^2. \qquad \text{[Prop. V.}$$

Ex. 1. CE parallel to the tangent at P meets SP, $S'P$ in E, E'. Prove that

(i) $PE = PE' = CA$.

(ii) $SE = SE'$.

(iii) the circle circumscribing the triangles CSE and $CS'E'$ are equal.

Ex. 2. The central perpendicular on the tangent at P meets SP produced in Q. Prove that the locus of Q is a circle. [Centre S. Radius $= CA$.]

Ex. 3. If from the centre of an ellipse lines be drawn parallel and perpendicular to the tangent at any point, they enclose a part of one of the focal distances of that point equal to the other.

Ex. 4. Given a focus and the length of the major axis, describe an ellipse touching two given straight lines.

Ex. 5. Given a focus, a tangent, and the eccentricity, the locus of the other focus is a circle. [Since $CS = e . CY$, the locus of the centre is obviously a circle.]

Ex. 6. Prove that the perimeter of the quadrilateral $SYY'S'$ is the greatest possible when YY' subtends a right angle at the centre.

Ex. 7. A line is drawn through S' parallel to SP meeting YS in O. Prove that the locus of Y is a circle.

Ex. 8. The right line drawn from either focus to the adjacent point of intersection of any tangent with the auxiliary circle is perpendicular to the tangent.

Ex. 9. If through any point Y on the auxiliary circle YP be drawn at right angles to SY, YP will be a tangent to the ellipse.

Ex. 10. If the vertex of a right angle moves on a fixed circle, and one leg passes through a fixed point, the other leg will always touch an ellipse. (Cf. Chap. I., Prop. XXIII., Ex. 4.)

Ex. 11. Given the major axis and a tangent, show that the directrix passes through a fixed point.

Ex. 12. The circle described on SP as diameter touches the auxiliary circle.

Ex. 13. Given a focus, a tangent, and the length of the major axis, the locus of the centre is a circle.

Ex. 14. Given the foci and a tangent, construct the ellipse.

Ex. 15. *Alternative Construction for Prop. XVIII.*

Let O be the external point. On OS as diameter describe a circle intersecting the auxiliary circle in Y and Y'. Then OY and OY' produced will be the tangents required.

Ex. 16. The right line drawn from the centre parallel to either focal radius vector of any point on an ellipse to meet the tangent at that point, is equal to the semi-axis major.

Ex. 17. Draw a tangent to an ellipse parallel to a given straight line.

Ex. 18. Two ellipses, whose axes are equal, each to each, are placed in the same plane, with their centres coincident and axes inclined to each other. Draw their common tangents. [The common tangents pass through the points in which the lines joining the foci of the curves meet the common auxiliary circle.]

Ex. 19. Given a focus, a tangent, and the length of the minor axis, the locus of the other focus is a straight line.

Ex. 20. If the rectangle under the perpendiculars from the fixed points on a right line be constant ($=k^2$), the line always touches an ellipse of which the fixed points are the foci, and the minor axis $=2k$.

Ex. 21 A chord of a circle, centre C and radius r, subtends a right angle at a fixed point O. Prove that it always touches an ellipse, of which C and O are the foci, and the square of the semi-axis minor $=r^2 \backsim CO^2$.

Ex. 22. If a second tangent to the ellipse intersect YPY' at right angles in O, prove that $OY \cdot OY' = CB^2$.

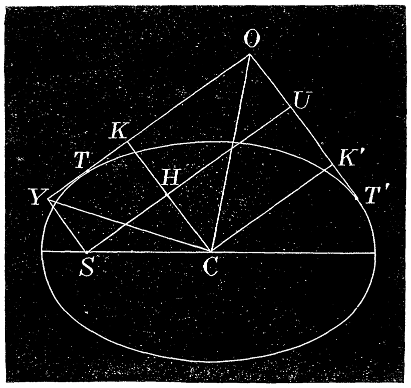
Hence, prove that $$CO^2 = CA^2 + CB^2.$$
(Cf. Prop. XXIV.)

* PROPOSITION XXIV.

The locus of the intersection of tangents to an ellipse which cut at right angles is a circle.

Let the tangents OT, OT' cut at right angles at O.

Draw SY, CK perpendicular to OT and SU, CK' perpendicular to OT'. Join CY, CU, CO. Let CK, SU intersect in H.



Now Y and U are on the auxiliary circle, [Prop. XXIII.

therefore $CY = CU = CA$.

Then $CO^2 = CK^2 + CK'^2$ [Euc. I. 47.

and $CY^2 = CK^2 + YK^2$,

therefore $CA^2 = CK^2 + SH^2$;

also $CU^2 = CK'^2 + UK'^2$,

therefore $CA^2 = CK'^2 + CH^2$,

therefore $2CA^2 = CK^2 + CK'^2 + SH^2 + HC^2$

$= CO^2 + CS^2$; [Euc. I. 47.

but $CS^2 = CA^2 - CB^2$, [Prop. V.

therefore $CO^2 = CA^2 + CB^2$.

Hence the locus of O is a circle described with the centre C and radius equal to AB.

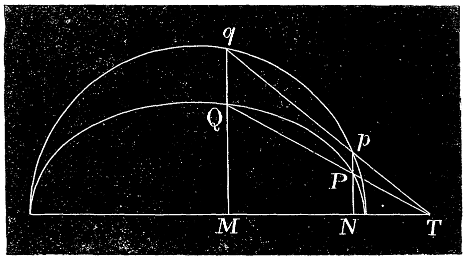
Note.—This circle is called the *Director Circle* of the ellipse.

Ex. 1. An ellipse slides between two fixed lines at right angles to each other; prove that the locus of its centre is an arc of a circle.

Ex. 2. Any rectangle circumscribing an ellipse is inscribed in the director circle.

PROPOSITION XXV.

Tangents at corresponding points of an ellipse and its auxiliary circle intersect on the major axis.



Let the ordinate pPN meet the ellipse in P and the auxiliary circle in the corresponding point p. Let qQM be any other ordinate.

Then, because

$$QM : qM = CB : CA$$
$$= PN : pN, \qquad \text{[Prop. XI.}$$

the straight line QP, qp produced meet the major axis in the same point T.

Now, if qQM be made to move parallel to itself so as to coincide with pPN, the points Q, P and q, p will coalesce, and the chords QPT and qpT will become tangents to the ellipse and the circle at P and p respectively.

Ex. 1. Deduce this proposition from the property $CN \,.\, CT = CA^2$. (Prop. XXII.)

Ex. 2. The tangent at p meets CB produced in K. Prove that $CK \,.\, PN = CA \,.\, CB$.

Ex. 3. Show that the locus of the intersection of the normals at P and p is a circle of which the radius is $CA+CB$. [If the normals intersect in O, and if PR be drawn parallel to the major axis to meet CO in R, then, by similar triangles, it may easily be shown that $OR=CA$, $CR=CB$.]

Ex. 4. OQ, OQ' are tangents to an ellipse; ON is drawn perpendicular to the axis. Prove that the tangents to the auxiliary circle at the corresponding points q, q' meet on ON.

If QQ' produced meet the major axis in T, prove also that

$$CN \cdot CT = CA^2.$$

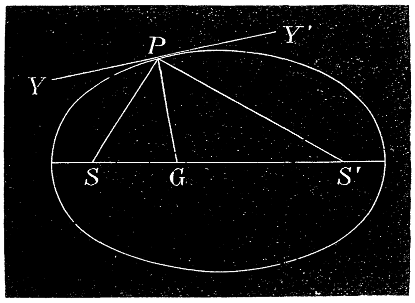
[For the second part, note that if ON meet the auxiliary circle in R, the tangent at R meets the major axis at the point where QQ', qq' meet it. Cf. also Prop. XXII., note, which is a limiting case.]

Ex. 5. In Ex. 4, if ON meets the ellipse in r, the tangent at r intersects the major axis in T.

PROPERTIES OF NORMALS.

PROPOSITION XXVI.

The normal at any point of an ellipse bisects the angle between the focal distances of the point.



Let the normal at the point P meet the major axis in G. Let YPY' be the tangent at P.

Then the angle SPY = the angle $S'PY'$. [Prop. XVII.

But the angles GPY, GPY' are equal, being right angles; [Def.

therefore the angle $SPG =$ the angle $S'PG$.

Ex. 1. If the tangent and normal at any point P meet the minor axis in t and g, then P, t, g, S, and S' lie on the same circle.

Ex. 2. Prove that the triangles SPG and gPS' are similar.

Ex. 3. If from g a perpendicular gK be drawn on SP or $S'P$, show that $PK = CA$.

Ex. 4. Prove that $SP \cdot S'P = PG \cdot Pg$. [The triangles PSg, $PS'G$ are similar. Ex. 1.]

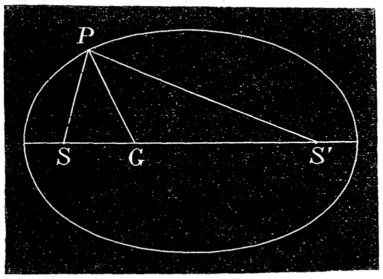
Ex. 5. No normal can pass through the centre, except it be at an end of one of the axes.

Ex. 6. The normal PG and the focal perpendiculars on the tangent at P are in harmonic progression.

Ex. 7. The circle described on PG as diameter cuts SP, $S'P$ in K and L. Prove that PG bisects KL at right angles.

*PROPOSITION XXVII.

If the normal at any point P of an ellipse meets the major axis in G, $SG = e \cdot SP$.



Join $S'P$.

Then, since PG bisects the angle SPS', [Prop. XXVI.

$$SG : S'G = SP : S'P; \quad \text{[Euc. VI. 3.}$$

therefore $$SG : SG + S'G = SP : SP + S'P,$$

or $$SG : SP = SG + S'G : SP + S'P.$$

But $SG + S'G = SS' = e . AA'$, [Prop. III.
and $SP + S'P = AA'$; [Prop. IV.
therefore $SG = e . SP$.

Ex. 1. Show how to draw the normal at any point without drawing the tangent.

Ex. 2. If PM be drawn perpendicular to the directrix, and MS meet the minor axis in g, show that Pg is the normal at P.

Ex. 3. A perpendicular is drawn from a fixed point M on the major axis of an ellipse, on the tangent at any point P. The locus of the intersection of this perpendicular with SP is a circle.

Ex. 4. If GE be perpendicular to SP, prove that PE is equal to half the latus rectum. [PSN and SEG are similar triangles; therefore $SE = e . SN$, $SP = e . NX$, so that $PE = e . SX$.]

Ex. 5. In Ex. 4, show that $GE = e . PN$.

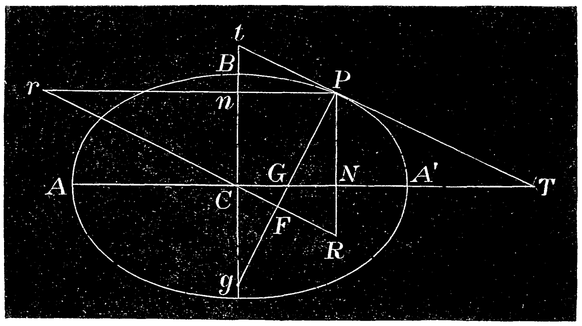
Ex. 6. Show that

$$PG^2 : SP . S'P = CB^2 : CA^2.$$

(Cf. Prop. XXVI., Ex. 4, and Prop. XXVIII.)

Proposition XXVIII.

The normal at any point of an ellipse, terminated by either axis, varies inversely as the central perpendicular on the tangent. ($PG . PF = CB^2 \quad Pg . PF = CA^2$.)



Let the normal at P meet the major axis in G and the minor axis in g; let the tangent at P meet them in T

and t respectively. Draw PN, Pn perpendicular to the major and minor axis, and let a straight line through the centre, drawn parallel to the tangent at P, meet PN, PG, and Pn produced, in R, F, and r respectively.

Then, since the angles at N and F are right angles, G, F, R, N lie on a circle; therefore

$$PG \,.\, PF = PN \,.\, PR \qquad \text{[Euc. III. 36.}$$
$$= Cn \,.\, Ct \qquad \text{[Euc. I. 34.}$$
$$= CB^2. \qquad \text{[Prop. XXII., Note.}$$

Again, since the angles at n and F are right angles, g, F, n, r lie on a circle; therefore

$$Pg \,.\, PF = Pn \,.\, Pr \qquad \text{[Euc. III. 36.}$$
$$= CN \,.\, CT \qquad \text{[Euc. I. 34.}$$
$$= CA^2. \qquad \text{[Prop. XXII., Note.}$$

Therefore both PG and Pg vary inversely as PF, which is equal to the central perpendicular upon the tangent at P.

Ex. 1. If CF meet the focal distances of P in E and E', prove that Pg subtends a right angle at E and E'. (See Prop. XXIII., Ex. 1.)

Ex. 2. If the circle through S, P, S' meets the minor axis in g on the side opposite to P, prove that Sg varies as PG.

Ex. 3. PQ is drawn at right angles to SP, meeting the diameter parallel to the tangent at P in Q. Prove that PQ varies inversely as PN.

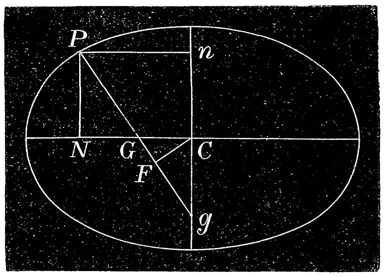
PROPOSITION XXIX.

If the normal at any point P on an ellipse meets the major axis in G, and PN be the ordinate to that axis,

(i) $GN : CN = CB^2 : CA^2$.

(ii) $CG = e^2 \,.\, CN$.

Let the normal meet the minor axis in g. Draw Pn perpendicular to the minor axis, and CF parallel to the tangent at P.



Then, because the triangles PNG and Png are similar,

$GN : CN = PG : Pg$ [Euc. VI. 2.

$= PG \,.\, PF : Pg \,.\, PF$

$= CB^2 : CA^2$; [Prop. XXVIII.

therefore $CN - GN : CN = CA^2 - CB^2 : CA^2$,

or $CG : CN = CS^2 : CA^2$. [Prop. V.

But $CS = e \,.\, CA$; [Prop. III.

therefore $CG = e^2 \,.\, CN$.

Ex. 1. In the figure of Prop. XXVIII., prove that :—

(i) $CG \,.\, CT = CS^2$.
(ii) $Cg \,.\, Ct = CS^2$.
(iii) $NG \,.\, CT = CB^2$.
(iv) Tg, tG intersect at right angles.

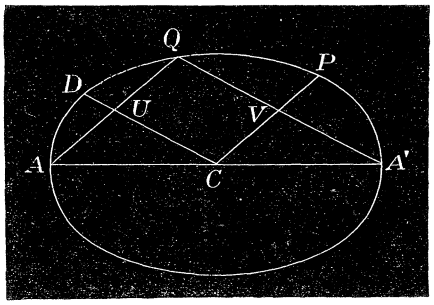
Ex. 2. Find a point P on the ellipse such that PG may bisect the angle between PC and PN.

Ex. 3. In the figure of Prop. XXVIII., prove that the rectangle under the focal perpendiculars on $PG = CF \,.\, PT$.

PROPERTIES OF CONJUGATE DIAMETERS.

Proposition XXX.

If one diameter of an ellipse bisects chords parallel to *a second, the second diameter bisects chords parallel to the first.*



Let CP bisect chords parallel to CD; then CD bisects chords parallel to CP.

Draw $A'Q$ parallel to CD, meeting CP in V; join AQ, meeting CD in U.

Then $A'Q$ is bisected in V and AA' in C; therefore CV is parallel to AQ. [Euc. VI. 2.

Again, since AA' is bisected in C, and CD is parallel to $A'Q$, AQ is bisected by CD. [Euc. VI. 2.
Therefore CD bisects all chords parallel to AQ, [Prop. X.
and therefore all chords parallel to CP.

Def. Two diameters so related that each bisects chords parallel to the other are called *Conjugate Diameters.*

Thus CP and CD are conjugate to each other; so also are the major and minor axes.

Ex. 1. If one diameter is conjugate to another, the first is parallel to the tangent at an extremity of the second. (Prop. XIV.)

Ex. 2. Given an ellipse and two conjugate diameters, show how to draw the tangent at any point.

If CP, CD be conjugate diameters, and QV is drawn parallel to CD, QV is the ordinate to CP. In CP produced take T, such that $CV . CT = CP^2$. QT is the tangent at Q. (Prop. XXII.)

Ex. 3. If CQ be conjugate to the normal at P, then CP is conjugate to the normal at Q.

Ex. 4. The focal perpendiculars upon CP and CD, when produced backwards, will intersect CD and CP on the directrix. (Apply Prop. XXIX., Ex. 2.)

Ex. 5. The focus is the orthocentre of the triangle formed by any two conjugate diameters and the directrix. (See Prop. X., Ex. 1.)

Ex. 6. Any diameter is a mean proportional between the focal chord parallel to it and the major axis. [The conjugate diameter CD will bisect the focal chord. Then apply Prop. XXII., and Prop. XXIII., Ex. 16.]

Ex. 7. The rectangle under the intercepts on any tangent between the curve and any two conjugate diameters, is equal to the square of the semi-diameter parallel to the tangent, and conversely.

Let the tangent at Q meet the conjugate semi-diameters CP, CD in T, T', and let CR be the semi-diameter parallel to TT'. Let the tangent at R parallel to CQ meet CD in t. Draw the ordinates QV, Rv with respect to CD, parallel to CP. Then

$$CV\,.\,CT' = Cv\,.\,Ct = CD^2. \qquad \text{[Prop. XXII.}$$

By similar triangles,

$$QT : CR = CV : Cv = Ct : CT' = CR : QT'.$$

Therefore $$QT\,.\,QT' = CR^2.$$

Ex. 8. Given in magnitude and position any two conjugate semi-diameters CP, CD of an ellipse, find the major and minor axes.

Produce CP to K, such that $CP\,.\,PK = CD^2$. Bisect CK in O, and let the line through O at right angles to CK meet the line through P parallel to CD in H. With centre H and radius HC, describe a circle cutting PH in T, T'; the circle will also pass through K. Then CT, CT' will coincide with the directions of the major and minor axes respectively.

For $PT\,.\,PT' = CP\,.\,PK = CD^2$; therefore CT, CT' are conjugate diameters (Ex. 7), and as they are at right angles, they must coincide with the directions of the major and minor axes. (Cf. Prop. XXXIII., Ex. 3; see also Miscellaneous Examples, 13, 14, 15, 16.)

To determine the magnitudes of the axes, observe that TPT' is the tangent at P, and apply Prop. XXII., note.

Ex. 9. PP' is a fixed line. Find the locus of a point Q which so moves that QV being drawn in a fixed direction to meet PP' in V, QV^2 is to $PV\,.\,P'V$ in a given ratio.

Bisect PP' in C, and through C draw CD in the fixed direction, such that CD^2 is to CP^2 in the given ratio. Then the locus of Q

will be the ellipse described with *CP* and *CD* as conjugate semi-diameters (Ex. 8). Apply Prop. XII., and cf. Prop. XXXII.

Note.—If $QV^2 = PV \,.\, P'V$, the semi-diameters *CP*, *CD* will be *equi-conjugate.* In this case the position of the major and minor axes may be at once determined, as they bisect the angles between the equiconjugate diameters. (See Prop. XXXI., Ex. 3.)

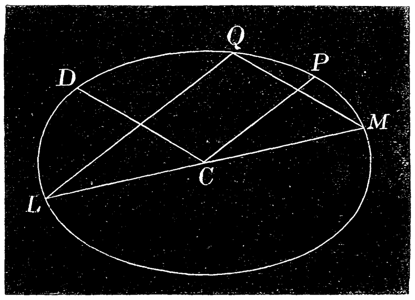
Ex. 10. A series of ellipses have their equiconjugate diameters of the same magnitude. One of these diameters is fixed and common, while the other varies. The tangents drawn from any point on the fixed diameter produced will touch the ellipses in points situated on a circle. (Apply Prop. XXII.)

Ex. 11. If *CN*, *CP* are the abscissa and ordinate of a point *P* on a circle whose centre is *C*, and *NQ* be taken equal to *NP*, and be inclined to it at a constant angle, the locus of *Q* is an ellipse.

Def. Chords which join any point on an ellipse to the extremities of a diameter are called *supplemental chords.*

PROPOSITION XXXI.

Supplemental chords of an ellipse are parallel to conjugate diameters.



Join any point *Q* on the ellipse to the extremities of a diameter *LCM.* Then *QL* and *QM* are supplemental chords.

Draw *CP*, *CD* parallel to *QL*, *QM* respectively; then they shall be conjugate diameters.

Because LM is bisected in C and CP is parallel to LQ, CP bisects MQ, [Euc. VI. 2.
and, therefore, all chords parallel to CD. [Prop. X.
Therefore CD bisects all chords parallel to CP, [Prop. XXX.
and is therefore conjugate to CP.

Ex. 1. Prove that for any assumed pair of conjugate diameters there can be drawn a pair of supplemental chords parallel to them.

Ex. 2. The diagonals of any parallelogram circumscribed to an ellipse are conjugate diameters. [The diagonals pass through the centre of the ellipse. Then see Note on *Tangent-Properties*, Ex. 1, 3.]

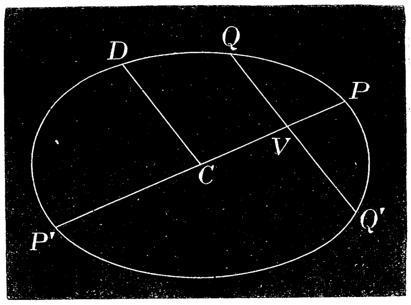
Ex. 3. The diagonals of the rectangle formed by the tangents at the extremities of the major and minor axes of an ellipse are equi-conjugate diameters.

Ex. 4. The tangent at any point Q on an ellipse meets the equi-conjugate diameters in T and T'. Prove that the triangles QCT and QCT' are as $CT^2 : CT'^2$. [Apply Prop. XXII.]

* Proposition XXXII.

The square of the ordinate of any point on an ellipse with respect to any diameter varies as the rectangle under the segments of the diameter made by the ordinate.

$$(QV^2 : PV \, . \, P'V = CD^2 : CP^2.)$$



Let QVQ' be a double ordinate with respect to the diameter PCP', meeting it in V. Let CD be the semi-diameter parallel to QV.

Now CP bisects QQ' and therefore all chords parallel to QV or CD. [Def. and Prop. X.

Therefore CD is conjugate to CP. [Def.

But $QV.Q'V:PV.P'V=CD^2:CP^2$. [Prop. XII.

and $QV=Q'V$.

Therefore $QV^2:PV.P'V=CD^2:CP^2$.

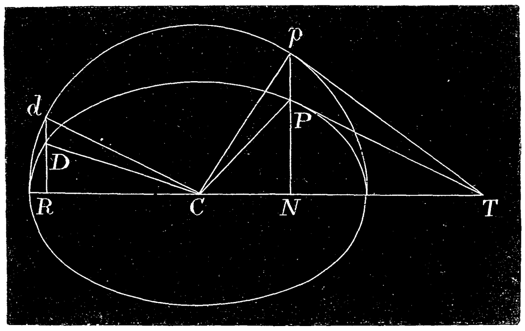
Ex. If QP, QP' meet CD, CP in M, N respectively, prove that $CM.CN=CD^2$.

Proposition XXXIII.

If CP, CD be two conjugate semi-diameters of an ellipse and ordinates PN, DR be drawn to the major axis, then

(i) $PN:CR=DR:CN=CB:CA$.

(ii) $CN^2+CR^2=CA^2$.



Let NP and RD produced meet the auxiliary circle in p and d. Join Cp, Cd, and let the tangents at P and p meet the major axis produced in T. [Prop. XXV.

Then, because PT is parallel to CD, [Props. X. and XIV. the triangles NPT and RDC are similar.

Therefore $NT : RC = PN : DR$; [Euc. VI. 4.
but $PN : DR = pN : dR$, [Prop. XI.
therefore $NT : RC = pN : dR$,
and the angles pNT and dRC are equal, being right angles. Therefore, the triangles NpT and RdC are similar. [Euc. VI. 6.
Therefore the angles pTN and dCR are equal.

Therefore pT is parallel to dC and the angle $dCP =$ the angle $CpT =$ a right angle.

Therefore the angle $pCN =$ the angle CdR, each being the complement of the angle dCR.

Therefore the two triangles pCN and dCR are equal in every respect. [Euc. I. 26.

Therefore $CR = pN$
and $PN : CR = PN : pN$
$= CB : CA$. [Prop. XI.
Similarly $DR : CN = CB : CA$.
Again,
$$CN^2 + CR^2 = CN^2 + pN^2$$
$$= Cp^2 = CA^2.$$

Ex. 1. If CQ be perpendicular to PT, prove that
$$CQ \,.\, QT : CT^2 = CN \,.\, PN : CD^2.$$

Ex. 2. If the normal at P meets the major and minor axes in G and g respectively, prove that
(i) $PG : CD = CB : CA$,
(ii) $Pg : CD = CA : CB$,
(iii) $PG \,.\, Pg = CD^2$.

Ex. 3. Prove that if two conjugate diameters be at right angles to each other, they must be the major and minor axes of the ellipse.

Ex. 4. Prove that
$$(SP - CA)^2 + (SD - CA)^2 = CS^2.$$

Ex. 5. If the tangent at the vertex A cut any two conjugate diameters in T and t, show that $AT \,.\, At = CB^2$.

Ex. 6. Apply Prop. XXII. to prove this proposition.

If the tangents at P and D meet the major axis in T and t, it may easily be shown from the relation

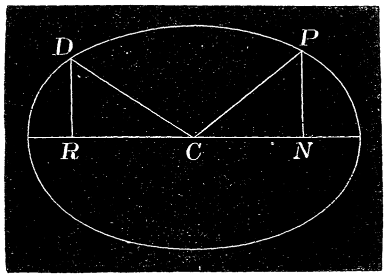
$$CR : CN = CT : Ct,$$

that $$CN^2 = CR \,.\, Rt = AR \,.\, A'R.$$

Then apply Prop. IX.

PROPOSITION XXXIV.

The sum of the squares of any two conjugate semi-diameters is constant. ($CP^2 + CD^2 = CA^2 + CB^2$.)



Let CP, CD be the conjugate semi-diameters, and let PN, DR be the ordinates to the major axis.

Then $PN : CR = CB : CA$. [Prop. XXXIII.

Therefore $PN^2 : CR^2 = CB^2 : CA^2$.

Similarly $DR^2 : CN^2 = CB^2 : CA^2$,

therefore $PN^2 + DR^2 : CN^2 + CR^2 = CB^2 : CA^2$;

but $CN^2 + CR^2 = CA^2$; [Prop. XXXIII.

therefore $PN^2 + DR^2 = CB^2$,

therefore $CP^2 + CD^2 = CA^2 + CB^2$. [Euc. I. 47.

Therefore, in the ellipse, the sum of the squares of any conjugate semi-diameters is constant, being equal to the sum of the squares of the semi-axis major and semi-axis minor.

Ex. 1. Find the greatest value of the sum of a pair of conjugate diameters. [The diameters must be equiconjugate.]

Ex. 2. If PG, DH be the normals at P and D, prove that PG^2+DH^2 is constant.

Ex. 3. Prove that $SP.S'P=CD^2$. [$SP+S'P=2CA$. Then square and substitute.]

Ex. 4. OP, OQ are tangents to an ellipse, and SQ is produced to meet the directrices in R, R'. Prove that

$$PR.PR' : QR.QR' = OP^2 : OQ^2.$$

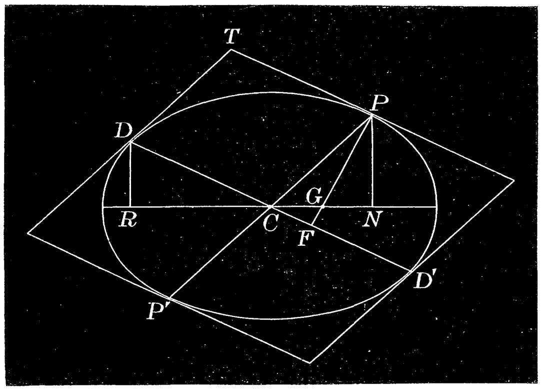
[If PM and QN be the ordinates, it can easily be shown that

$$\frac{PR.PR'}{QR.QR'}=\frac{MX.MX'}{NX.NX'}=\frac{SP.S'P}{SQ.S'Q}.$$

Then apply Ex. 3 and Note on *Tangent-Properties*, Ex. I., 1.]

* Proposition XXXV.

The area of the parallelogram formed by the tangents at the extremities of a pair of conjugate diameters is constant. ($CD.PF=CA.CB$.)



The tangents at the extremities of two conjugate diameters PCP' and DCD' will evidently form a parallelogram, [Prop. XIV.
the area of which is four times that of the parallelogram $CDTP$, where T is the intersection of the tangents at P and D.

Let the normal at P meet the major axis in G and DCD' in F. Draw the ordinates PN and DR to the major axis.

Then, since the angles at N and F are right angles, the angle GPN = the angle GCF = the angle DCR.

[Euc. I. 15 and I. 32.

Therefore the two right-angled triangles GPN and DCR are similar.

Therefore $PG : CD = PN : CR$

$= CB : CA$, [Prop. XXXIII.

therefore $PG \,.\, PF : CD \,.\, PF = CB^2 : CA \,.\, CB$;

but $PG \,.\, PF = CB^2$, [Prop. XXVIII.

therefore $CD \,.\, PF = CA \,.\, CB$.

Again, the area of the parallelogram $CDTP$

$= CD \,.\, PF = CA \,.\, CB =$ constant,

which proves the proposition.

Ex. 1. Find the least value of the sum of a pair of conjugate diameters. [The diameters are the major and the minor axis. Cf. Prop. XXXIV., Ex. 1.]

Ex. 2. Prove that the parallelogram formed by the tangents at the extremities of a pair of conjugate diameters is the least that can be circumscribed about the ellipse.

Ex. 3. If PG meets the minor axis in g, prove that

$$PG \,.\, Pg = CD^2.$$

(Prop. XXVIII. Cf. Prop. XXXIII., Ex. 2.)

Ex. 4. If SY be the perpendicular upon the tangent at P, prove that

$$SP : SY = CD : CB.$$

[In the figure of Prop. XXIII.,

$$\frac{SP}{SY} = \frac{S'P}{S'Y'} = \frac{SP + S'P}{SY + S'Y'} = \frac{CA}{CK},$$

where CK is the central perpendicular upon the tangent at P.

Therefore $$\frac{SP}{SY} = \frac{CA}{PF} = \frac{CD}{CB}. \Big]$$

Ex. 5. Prove that $SP \,.\, S'P = CD^2$. [From Ex. 4

$$\frac{SP \,.\, S'P}{SY \,.\, S'Y'} = \frac{CD^2}{CB^2}.$$

Then apply Prop. XXIII. Cf. also Prop. XXXIV., Ex. 3, and Prop. XXXIII., Ex. 2, along with Prop. XXVI., Ex. 4.]

Ex. 6. If the tangent at P meet the minor axis in T, prove that the areas of the triangles SPS', STS' are as $CD^2 : ST^2$. [Cf. Prop. XXVI., Ex. 1.]

Ex. 7. If DQ be drawn parallel to SP and CQ perpendicular to DQ, prove that $CQ = CB$. (See Ex. 4.)

Ex. 8. The tangents drawn from D to the circle on the minor axis as diameter are parallel to the focal distances of P. (See Ex. 4.)

Ex. 9. If on the normal at P, PQ be taken equal to the semi-conjugate diameter CD, the locus of Q is a circle whose centre is C and radius equal to $CA - CB$. [Apply Prop. XXXIV.]

MISCELLANEOUS EXAMPLES ON THE ELLIPSE.

1. Find the locus of the point of intersection of any tangent to an ellipse, with the line drawn from the focus making a constant angle with the tangent.

[A circle. Cf. Prop. XXIII. Observe that if the vertex of a triangle of a given species be fixed, while one base angle moves along a fixed circle, the locus of the other base angle is a circle.]

2. The line drawn parallel to the axis through the intersection of normals at the extremities of a focal chord, bisects the chord.

3. S, S' are the foci of an ellipse; $S'R$ is drawn equal to AA'; the line bisecting RS at right angles touches the ellipse. (Newton, Book I., Prop. XVII.)

4. Given a focus, the length of the major axis and two points on the curve, to construct it. (Apply Prop IV. Newton, Book I., Prop. XVIII.)

5. Given a focus, the eccentricity, and two tangents, to construct the curve. (Apply Prop. XXIII., Ex. 5. Newton, Book I., Prop. XX.)

6. Given a focus, the eccentricity and two points

on the curve, to construct it. (Newton, Book I., Prop. XX.)

[The directrix touches the two circles having their centres at the given points, and radii equal to e times their focal distances.]

7. Given a focus and the eccentricity, to describe an ellipse touching a given line at a given point. (Newton, Book I., Prop. XX.)

[Let S be the given focus, and P the given point on the tangent YPY'. (Fig. Prop. XXIII.) Draw SY at right angles to PY, and produce it to R, so that $YR = YS$. Divide SR internally and externally at the points K, L in the ratio $SA : AX$; the circle on KL as diameter meets RP in S'.]

8. The rectangle under the perpendiculars let fall from any point on an ellipse on two opposite sides of an inscribed quadrilateral is in a constant ratio to the rectangle under the perpendiculars let fall on the other two sides.

[The proposition holds if instead of perpendiculars on the sides, lines are drawn making a constant angle with them. Newton, Book I., Lemmas XVII.-XIX.]

9. The rectangle under the perpendiculars let fall from any point on an ellipse on two fixed tangents is in a constant ratio to the square of the perpendicular on their chord of contact.

10. If two fixed tangents to an ellipse be cut by a diameter parallel to their chord of contact and by a third variable tangent, the rectangle under the segments of the two fixed tangents, intercepted between the diameter and the variable tangent, is constant.

11. The right line joining the middle points of the diagonals of a quadrilateral circumscribing an ellipse will pass through the centre. (Apply Ex. 10 and Prop. XXI., Ex. 5.)

12. If a quadrilateral be circumscribed to an ellipse the diagonals will intersect on the chord of contact of the sides.

13. Given two conjugate diameters in magnitude and position to construct the ellipse.

[Through the extremities P, P', D, D' of the given conjugate diameters PCP', DCD', draw lines parallel to them, forming the parallelogram $EFGH$. Divide the half side DE into any number of equal parts at R', R'', etc. Divide DC into the same number of equal parts at r', r'', etc. The intersection of PR' and $P'r'$ determines a point on the ellipse.]

14. Given two conjugate semi-diameters in magnitude and position, determine the axes.

[Let CP, CD be the conjugate semi-diameters. Draw PR perpendicular to CD, and on PR take PQ, PQ' on opposite sides of P, each equal to CD; then the axes are in direction the bisections of the angle QCQ', while their lengths are the sum and difference of CQ, CQ'.]

15. Given two conjugate semi-diameters in magnitude and position, determine the axes.

[Let CP, CD be the conjugate semi-diameters. Draw PR perpendicular to CD, and on it take $PQ=CD$. On CQ as diameter, describe a circle, and let O be its centre. Join OP, cutting the circle in E and F; join CE, CF, and take on CE, CF, $CA=FP$, $CB=EP$. Then CA, CB are the semi-axes sought.]

16. Given two conjugate semi-diameters CP, CD, with centre C and radius CP describe a circle, and let KK' be its diameter at right angles to CP; then will the axes of the ellipse be equal to $KD \pm K'D$, and parallel to the bisectors of the angle KDK'.

17. Any diameter of an ellipse varies inversely as the perpendicular focal chord of its auxiliary circle.

18. If any rectangle circumscribe an ellipse the perimeter of the parallelogram formed by joining the points

of contact is twice the diameter of the director circle. (Prop. XXIV.)

19. Given a focus, the length of the major axis, and that the second focus lies on a fixed straight line, prove that the ellipse touches two fixed parabolas having the given focus for focus.

20. Two given ellipses in the same plane have a common focus, and one revolves about the common focus while the other remains fixed; the locus of the point of intersection of their common tangents is a circle.

[If H be the second focus of the fixed ellipse, K of the revolving ellipse, and b_1, b_2 their semi-minor axes,

$$HT : KT = b_1^2 : b_2^2,$$

where T is the point whose locus is sought.]

21. TQ, TQ' are tangents to an ellipse; CQ, CQ', QQ', CT are joined; QQ' and CT intersect in V. Prove that the area of the triangle QCQ' varies inversely as

$$\left(\frac{CV}{TV}\right)^{\frac{1}{2}} + \left(\frac{TV}{CV}\right)^{\frac{1}{2}}.$$

22. SY, $S'Y'$ are perpendiculars on the tangent at P. Perpendiculars from Y, Y' on the major axis cut the circles of which SP, $S'P$ are diameters in I, J respectively. Prove that IS, JS', and BC produced meet in the same point.

23. An ellipse touches two given lines OP, OQ in P and Q, and has one focus on the line PQ. Find the other focus and the directrices.

24. S, S' are the foci of an ellipse; SY is perpendicular on the tangent at P. Prove that $S'Y$ bisects the normal at P.

25. CP, CD are two conjugate semi-diameters of an

ellipse; *Rr* is a tangent parallel to *PD*; a straight line *CIJ* cuts at a given angle *PD*, *Rr* in *I*, *J*. Prove that the loci of *I* and *J* are similar curves. [It can easily be shown that $CI^2 : CJ^2 = 1 : 2$.]

26. A system of parallelograms is inscribed in an ellipse whose sides are parallel to the equiconjugate diameters. Prove that the sum of the squares on the sides is constant.

27. *OP*, *OQ* are tangents to an ellipse; *CU*, *CV* are the parallel semi-diameters. Prove that

$$OP \,.\, OQ + CU \,.\, CV = OS \,.\, OS'.$$

28. *P*, *Q* are points on two confocal ellipses at which the line joining the common foci subtends equal angles. Prove that the tangents at *P*, *Q* include an angle equal to that subtended by *PQ* at either focus.

29. The foci of a given ellipse *A* lie on an ellipse *B* the extremities of a diameter of *A* being the foci of *B*. Prove that the eccentricity of *B* varies as the diameter of *A*.

30. Normals at the extremities *P* and *D* of two conjugate semi-diameters meet in *K*. Prove that *CK* is perpendicular to *PD*.

31. If *CP*, *CP'* be semi-diameters of an ellipse at right angles to each other, prove that

$$\frac{1}{CP^2} + \frac{1}{CP'^2}$$

is constant.

32. Having given the auxiliary circle of an ellipse and a tangent to the ellipse touching the ellipse at a given point, find the foci.

33. Find the locus of the centres of circles cutting a given ellipse orthogonally.

34. An ellipse is inscribed in a given triangle. If one of the foci is known, show how to find the ellipse and its points of contact with the sides of the triangle.

35. Two fixed points Q, R and a variable point P are taken on an ellipse; the locus of the orthocentre of the triangle PQR is an ellipse.

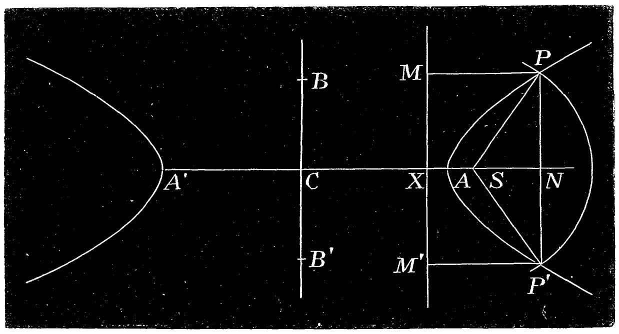
CHAPTER III.

THE HYPERBOLA.

DESCRIPTION OF THE CURVE.

PROPOSITION I.

Given the focus, directrix, and eccentricity of a hyperbola to determine any number of points on it.



Let S be the focus, MXM' the directrix, and e the eccentricity.

Through S draw SX perpendicular to the directrix. Divide SX in A so that

$$SA = e \, . \, AX.$$

Also, in SX produced,* take A' so that

$$SA'=e\,.\,A'X.$$

Then A and A' are points on the hyperbola and are its vertices.

Take any point N on $A'A$ produced. Through N draw PNP' perpendicular to AA'. With centre S and radius equal to $e\,.\,XN$, describe a circle cutting PNP' in P and P'. Then P and P' shall be points on the hyperbola.

Draw PM, $P'M'$ perpendicular to the directrix.

Then $$SP=e\,.\,XN \qquad \text{[Const.}$$
$$=e\,.\,PN,$$
and $$SP'=e\,.\,XN$$
$$=e\,.\,P'M'.$$

Therefore P and P' are points on the hyperbola.

In like manner, by taking any other point on $A'A$ produced, a series of points on the curve may be determined lying on the right hand side of the directrix.

Again, if N be taken on AA' produced, another series of points on the curve may be determined lying on the left hand side of the directrix.

Def. The length of the axis intercepted between the vertices (A, A') of the hyperbola is called the *transverse axis*.

Def. The middle point (C) of the transverse axis is called the *centre* of the hyperbola.

Def. A straight line BCB' passing through the centre and perpendicular to the transverse axis, such that

$$CB^2=CB'^2=CS^2-CA^2=SA\,.\,SA'$$

is called the *conjugate axis*.

---

* Since e is greater than unity, it is clear that A will lie between S and X, and A' without SX on the side remote from S.

The conjugate axis, unlike the minor axis of the ellipse, does not meet the curve at all. (See Ex. 3 below.) Its utility in establishing properties of the hyperbola will appear later on.

Ex. 1. The hyperbola is symmetrical with respect to its axis.

Corresponding to any point N on the line $A'A$ produced, we get two points P and P' such that the chord PP' is bisected at right angles by the axis $A'A$. [Def.

Ex. 2. Any two right lines drawn from any point on the axis to the curve on opposite sides of the axis, and equally inclined to it, are equal, and conversely.

Ex. 3. Show that the hyperbola lies wholly outside the lines drawn through A and A' at right angles to the axis.

In order that the circle may intersect the line PNP', the point N must be so situated that SN may not be greater than the radius of the circle SP, that is, $e \cdot XN$. It may be shown that this is the case only when N does not lie between A and A'.

Ex. 4. Hence, the hyperbola consists of *two distinct branches* lying on opposite sides of the lines drawn through the vertices at right angles to the axis.

Ex. 5. There is no limit to the distance to which each branch of the hyperbola may extend on both sides of the axis, so that the hyperbola consists of two *infinite* branches.

It is obvious that the point N may be taken anywhere on the axis beyond A and A'.

Note.—It will be remembered that the parabola consists of one infinite branch (Chap. I., Prop. I., Ex. 9) and that the ellipse is a closed oval (Chap. II., Prop. I., Ex. 6).

Ex. 6. In any conic, if PR be drawn to the directrix parallel to a fixed straight line, the ratio $SP : PR$ is constant.

Ex. 7. If an ellipse, a parabola, and a hyperbola have the same focus and directrix, the parabola will lie entirely outside the ellipse and inside the hyperbola. (Cf. Chap. I., Prop. I., Ex. 6 and 7.)

Ex. 8. Prove that the locus of a point of trisection of an arc of a circle described on a given base is a hyperbola.

Ex. 9. If a circle touches the transverse axis at the focus, and passes through one end of the conjugate axis, the portion of the conjugate axis intercepted $= CA^2/CB$.

Ex. 10. Prove that the locus of the point of intersection of two tangents to a parabola which cut at a constant angle is a hyperbola.

Let OP, OQ be two tangents to a parabola, cutting at a constant angle α. Draw OI, OU perpendicular to the directrix and SP; then $OI = SU$ (Chap. I., Prop. XIII.), and

$$OS : OI = OS' : SU,$$

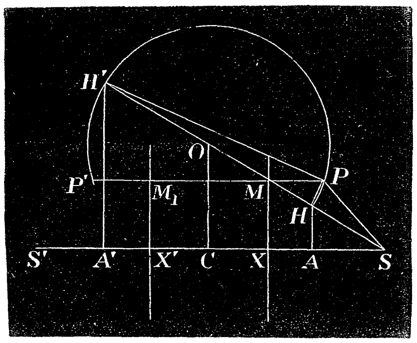
which is a constant ratio greater than unity since $\angle OSP=\pi-\alpha$. (Chap. I., Prop. XIX.) The locus of O is, therefore, a hyperbola having the same focus and directrix as the parabola.

Ex. 11. P is any point on a given hyperbola ($e=2$). D is taken on the axis such that $SD=SA'$. If $A'P$ meets the latus rectum in K, find the locus of the intersection of DK and SP. [The circle on $A'D$ as diameter.]

Ex. 12. The angular point A of a triangle ABC is fixed, and the angle A is given, while the points B and C move on a fixed right line. Find the locus of the centre of the circumscribing circle of the triangle. [A hyperbola of which A is the focus and BC the directrix.]

Proposition II.

The hyperbola is symmetrical with respect to the conjugate axis and has a second focus (S') and directrix.



Let S be the given focus and MX the given directrix.

Take any point M on the directrix and through the vertices A and A' draw AH, $A'H'$ at right angles to AA', meeting the straight line through M and S at H and H' respectively.

Describe a circle on HH' as diameter, and through M draw PMP' parallel to AA', to meet the circle in P and P'. Then P and P' shall be points on the hyperbola.

For $SH : HM = SA : AX$

$= e,$

and $SH' : MH' = SA' : XA'$

$= e,$

therefore $SH : HM = SH' : MH',$

and the angle HPH' is a right angle; therefore PH bisects the angle SPM.

Therefore $SP : PM = SH : HM$

$= SA : AX$

$= e.$

Therefore P is a point on the hyperbola.

Similarly it may be shown that P' is a point on the hyperbola.

Again, the straight line drawn through O, the centre of the circle, at right angles to AA', will bisect both AA' and PP' at right angles, and will therefore coincide with the conjugate axis in position.

The hyperbola is therefore symmetrical with respect to the conjugate axis.

Hence the two branches of the hyperbola, lying on opposite sides of the conjugate axis, are such that each is the exact reflexion of the other. Therefore, if $A'S'$ be measured off $= AS$ and $A'X' = AX$, and $X'M_1$ be drawn at right angles to $X'S$, the curve could be equally well described with S' as focus and $X'M_1$ as directrix. The hyperbola has therefore a second focus S' and a second directrix $X'M_1$.

Ex. Every chord drawn through the centre C and terminated by the two branches is bisected at that point. [From the symmetry of the figure.]

From this property the point C is called the *centre* of the curve.

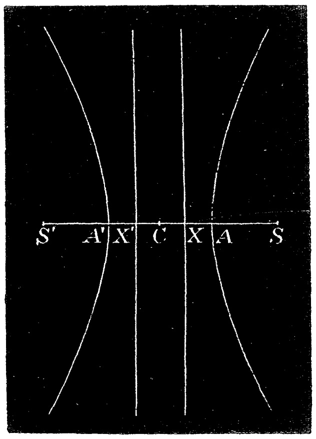
PROPERTIES OF CHORDS AND SEGMENTS OF CHORDS.

Proposition III.

In the hyperbola $CA = e\,.\,CX$ ....(i.)

$CS = e\,.\,CA$ ....(ii.)

$CS\,.\,CX = CA^2$ ....(iii.)



We have from the definition

$$SA = e\,.\,AX,$$

$$SA' = e\,.\,A'X = e\,.\,AX'.$$

Therefore, by subtraction,

$$AA' = e(AX' - AX)$$

$$= e\,.\,XX'.$$

Therefore $CA = e\,.\,CX$ ....(i.)

By addition $SS' = e\,.\,(AX + A'X)$

$$= e\,.\,AA'.$$

Therefore $CS = e\,.\,CA$ ....(ii.)

Therefore $CS\,.\,CX = CA^2$ ....(iii.)

Ex. 1. Given the transverse and the conjugate axis, find the focus and the directrix.

Ex. 2. Prove that $e^2 = 1 + \frac{CB^2}{CA^2}$.

Ex. 3. If the line through B parallel to the transverse axis meet the latus rectum in D, then will the triangles SCD, SXD be similar.

Ex. 4. Prove that

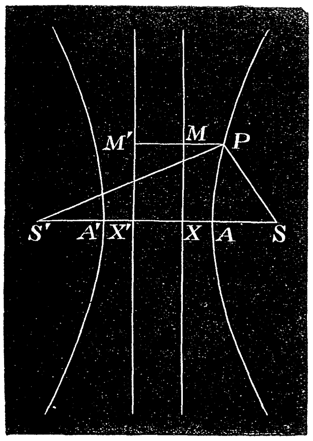
$$SX^2 : AX \cdot A'X = CB^2 : CA^2.$$

Ex. 5. If any line through the centre meet the perpendicular through A to the transverse axis in O and the directrix in E, then AE is parallel to SO.

Ex. 6. In Prop. I., Ex. 8, find the distance between the centres of the two hyperbolas which are the loci of the points of trisection of an arc of a circle described on a given base. [One-third of the given base.]

PROPOSITION IV.

The difference of the focal distances of any point on a hyperbola is constant and equal to the transverse axis.



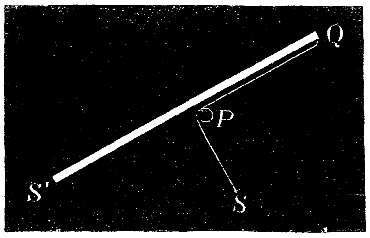
Let P be any point on the hyperbola. Join PS, PS', and through P draw PMM' perpendicular to the directrices.

Then $SP = e . PM,$

and $S'P = e . PM'.$

Therefore
$$\begin{aligned} S'P - SP &= e(PM' - PM) \\ &= e . MM' \\ &= e . XX' \\ &= AA'. \end{aligned}$$ [Prop. III.

Ex. 1. Show how to construct the hyperbola mechanically.



First Method.—Suppose a bar SQ, length r, to revolve round its extremity S' which is fixed. Then if a string of given length l, attached to the bar at Q and also to a fixed point S, be always kept stretched by means of a pencil at P pressed against it (the part QP of the string being in contact with the rod), the pencil will trace out a hyperbola with foci at S and S', and the transverse axis equal to $(r - l)$. For

$$S'P + PQ = r$$

and $$SP + PQ = l,$$

$\therefore$ $$S'P - SP = r - l = \text{constant}.$$

It should be observed that l must be less than r and greater than $r - SS'$.

In the same manner, by making the bar revolve round S as centre, the other branch of the hyperbola may be described. The other branch may also be described by taking the string longer than the rod by the length $(r - l)$.

Second Method.—Suppose two equal thin circular discs A and B attached to each other, to rotate in the same direction round an axis through their common centre; and suppose the two ends of a fine string (which is wrapped round the discs and passing through small rings at C and D in the plane of the discs, is kept stretched by the point of a pencil at P) to be wound off from the two discs. The curve traced by P will have the property $CP - DP = \text{constant}$, and will, therefore, be a hyperbola.

Ex. 2. Given the foci and the transverse axis to determine any number of points on the curve.

Describe a circle with centre S and any radius r; describe a circle with centre S' and radius $= r + AA'$. The two circles intersect in points on the curve.

Ex. 3. Given a focus, a tangent, and a point on an ellipse, prove that the locus of the other focus is a hyperbola. [The foci will be the given point and the image of the focus in the tangent. Chap. II., Prop. XXIII.]

Ex. 4. Given a focus, a tangent, and two points on an ellipse to construct the curve. (Newton, Book I., Prop. XXI.)

Ex. 5. Given a focus, two tangents, and a point on an ellipse to construct the curve. (Newton, Book I., Prop. XXI.)

Ex. 6. Given a focus, the eccentricity, a tangent, and a point on an ellipse to construct the curve. (Apply Chap. II., Prop. XXIII., Ex. 5. Newton, Book I., Prop. XX.)

Ex. 7. The difference of the focal distances of any point is greater than, equal to, or less than the transverse axis, according as the point is within, upon, or without the hyperbola, and conversely.

Ex. 8. The locus of the centre of a circle which touches two fixed circles is an ellipse or a hyperbola. (Cf. Chap. II., Prop. IV., Ex. 4.)

Ex. 9. Given one focus of an ellipse and two points on the curve, the locus of the other focus is a hyperbola.

Ex. 10. A parabola passes through two fixed points, and has its axis parallel to a given line; prove that the locus of its focus is a hyperbola.

Ex. 11. Given the base of a triangle and its point of contact with the inscribed circle, show that the locus of its vertex is a hyperbola.

Ex. 12. Find the locus of the intersection of the tangents from two given points A and B to all circles touching AB at a given point C.

[An ellipse when C is outside A and B; a hyperbola when C is between A and B, except when $CA = CB$, in which case the locus is a right line.]

Ex. 13. An ellipse and a hyperbola having the same foci intersect in P. If CA, Ca be their semi-axes major respectively and PN the ordinate of P, show that

$$CA : CS = CN : Ca.$$

Ex. 14. P is any point on an ellipse, of which CA, CB are the semi-axes; CD is the semi-diameter conjugate to CP; Cb is the semi-conjugate axis of the confocal hyperbola through P. Prove that

$$CB^2 + Cb^2 = CD^2.$$

Let $Ca =$ semi-transverse axis.

Then
$$\begin{aligned} Cb^2 &= CS^2 - Ca^2 = CS^2 - \tfrac{1}{4}(SP - S'P)^2 \\ &= CS^2 - \tfrac{1}{4}\{(SP + S'P)^2 - 4SP \,.\, S'P\} \\ &= CD^2 - CB^2. \end{aligned}$$
[Chap. II., Prop. XXXV., Ex. 5.

Ex. 15. SY, $S'Y'$ are the focal perpendiculars on the tangent at any point P of an ellipse. Prove that $PY.PY'$ is equal to the square on the semi-conjugate axis of the confocal hyperbola through P.

$$\left[\frac{SP}{PY}=\frac{S'P}{PY'}=\frac{CD}{\sqrt{PY.PY'}}.\right.$$

$$\frac{SY}{PY}=\frac{SY'}{PY'}=\frac{CB}{\sqrt{PY.PY'}}.$$

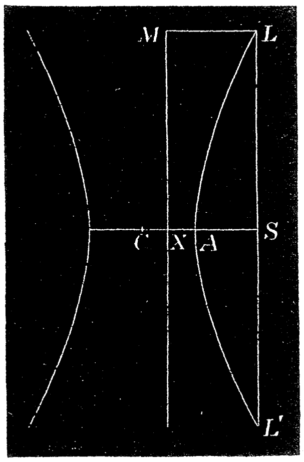
Apply Ex. 14. Cf. Prop. XXI., Ex. 8.]

Ex. 16. Two adjacent sides of a quadrilateral are given in magnitude and position; if a circle can be inscribed on the quadrilateral, the locus of the intersection of the other two sides is a hyperbola.

Ex. 17. Prove that the circle in Prop. I., Ex. 12, always touches a fixed circle. [Centre is second focus of the hyperbola, radius = transverse axis.]

* PROPOSITION V.

The latus rectum of a hyperbola is a third proportional to the transverse and conjugate axes. $\left(SL=\frac{CB^2}{CA}.\right)$



Let LSL' be the latus rectum. Draw LM perpendicular to the directrix.

Then $CS = e \, . \, CA.$ [Prop. III.

$SL = e \, . \, LM$ [Def.

$= e \, . \, SX.$

Therefore $SL \, . \, CA = CS \, . \, SX$

$= CS(CS - CX)$

$= CS^2 - CS \, . \, CX$

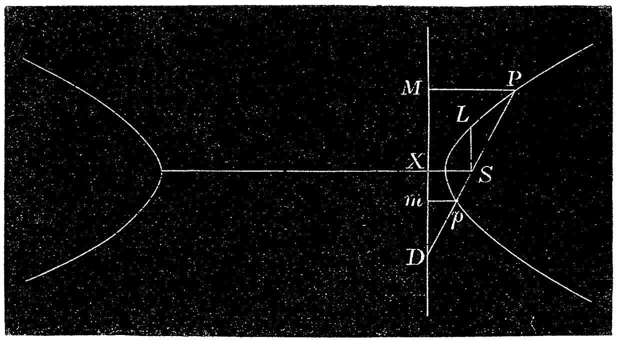
$= CS^2 - CA^2$ [Prop. III.

$= CB^2.$ [Def.

Ex. Prove this proposition by means of Prop. III., Ex. 4.

* PROPOSITION VI.

Any focal chord of a hyperbola is divided harmonically by the focus and directrix; and focal chords are to one another as the rectangles contained by their segments.



Produce the focal chord PSp to meet the directrix in D, and draw PM and pm perpendicular to the directrix.

Then $PD : pD = PM : pm;$

but $PS = e \, . \, PM,$

and $pS = e \, . \, pm;$ [Def.

therefore $PD : pD = PS : pS.$

Hence Pp is divided harmonically in S and D.

Again, PD, SD, and pD being in harmonic progression, PM, SX, and pm are also in harmonic progression. But

$$SP : PM = SL : SX = Sp : pm = e;$$

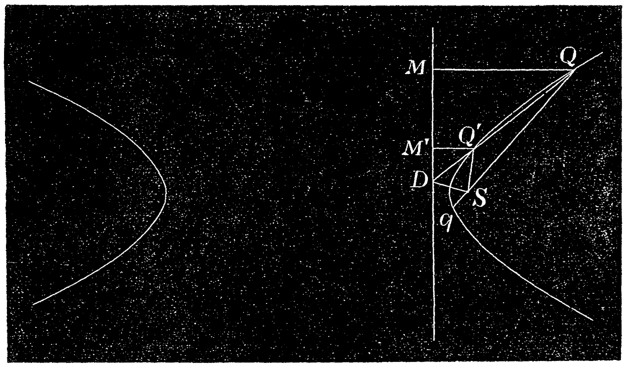
therefore SP, SL, and Sp are also in harmonic progression. Therefore

$$SL = \frac{2SP \,.\, Sp}{SP + Sp} = \frac{2SP \,.\, Sp}{Pp};$$

therefore the focal chord Pp varies as $SP \,.\, Sp$.

Proposition VII.

If any chord QQ' of a hyperbola intersects the directrix in D, SD bisects the angle between SQ and SQ'.



First, let Q and Q' be on the same branch of the hyperbola.

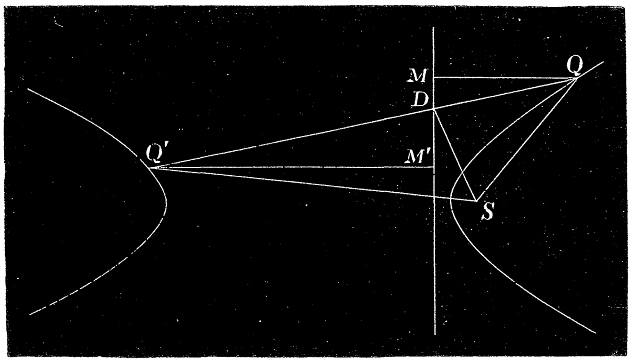
Draw QM, $Q'M'$ perpendicular to the directrix.

Then, by similar triangles,

$$\begin{aligned} QD : Q'D &= QM : Q'M' \\ &= SQ : SQ'. \end{aligned}$$

Therefore SD bisects the *exterior* angle $Q'Sq$. [Euc. VI. A.

Secondly, let Q, Q' be on opposite branches of the hyperbola; then it may be similarly shown that SD bisects the *interior* angle QSQ'. [Euc. VI. 3.



Ex. 1. Prove that a straight line can cut a hyperbola in two points only. (Cf. Chap. I., Prop. X., Ex. 8; Chap. II., Prop. VIII., Ex. 9.)

Ex. 2. If two points Q, Q' on a hyperbola be joined with a third variable point O on the curve, the segment qq' intercepted on either directrix by the chords QO and $Q'O$ produced, subtends a constant angle at the corresponding focus.

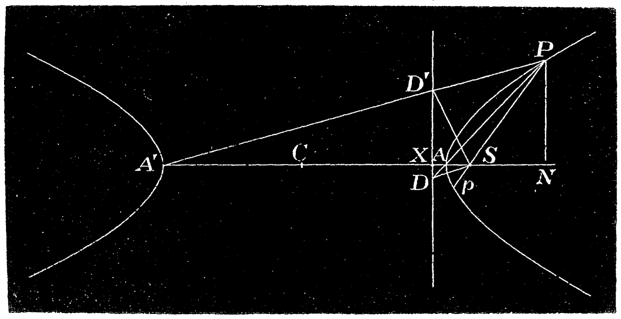
Ex. 3. Given the focus and three points on a hyperbola, find the directrix and the axis.

Proposition VIII.

The square of the ordinate of any point on a hyperbola varies as the rectangle under the segments of the axis produced, made by the ordinate.

$$(PN^2 : AN \,.\, A'N = CB^2 : CA^2.)$$

Let PN be the ordinate of any point P on the hyperbola. Let PA, PA', produced if necessary, meet the directrix in D and D'. Join SP, SD, SD', and produce PS to meet the curve in p.



Then, from the similar triangles PAN and DAX,

$$PN : AN = DX : AX.$$

Also from the similar triangles $PA'N$ and $D'A'X$,

$$PN : A'N = D'X : A'X;$$

therefore $PN^2 : AN . A'N = DX . D'X : AX . A'X.$

Again, SD and SD' bisect the angles pSX and PSX respectively; [Prop. VII.
therefore the angle DSD' is a right angle, and

$$DX . D'X = SX^2.$$ [Euc. VI. 8.

Therefore $PN^2 : AN . A'N = SX^2 : AX . A'X.$

But the ratio $SX^2 : AX . A'X$ is constant; therefore the ratio $PN^2 : AN . A'N$ has the same value for all positions of P.

To determine this constant value we have

$$SA : AX = CS : CA;$$ [Prop. III.

therefore $SX : AX = CS + CA : CA.$

Similarly $SX : A'X = CS - CA : CA;$

therefore $SX^2 : AX . A'X = CS^2 - CA^2 : CA^2$

$= CB^2 : CA^2$; [Def.

therefore $PN^2 : AN . A'N = CB^2 : CA^2$.

Ex. 1. Prove that

$$PN^2 : CN^2 - CA^2 = CB^2 : CA^2.$$

Ex. 2. Having shown that

$$PN^2 : AN . A'N = SX^2 : AX . A'X,$$

apply Prop. V. to complete the proof. [Make P coincide with the extremity L of the latus rectum.]

Ex. 3. Prove that

$$\frac{CN^2}{CA^2} - \frac{PN^2}{CB^2} = 1.$$

Ex. 4. NQ parallel to AB meets the conjugate axis in Q. Show that $QB . QB' = PN^2$.

Ex. 5. Q is a point on the curve; AQ, $A'Q$ meet PN in D and E; prove that $DN . EN = PN^2$.

Ex. 6. If a point P moves such that $PN^2 : AN . A'N$ in a constant ratio, PN being the distance of P from the line joining two fixed points A, A', and N falling outside AA'; the locus of P is a hyperbola of which AA' is an axis.

Ex. 7. PNP' is a double ordinate of an ellipse; show that the locus of intersection of AP' and $A'P$ is a hyperbola.

Ex. 8. A circle is described through A, A' and P. If NP meets the circle again in Q, the locus of Q is a hyperbola.

Ex. 9. NQ is a tangent to the circle on AA' as diameter; PM is drawn parallel to CQ, meeting AA' in M; show that $MN = CB$. [The triangles PMN, QCN are similar.]

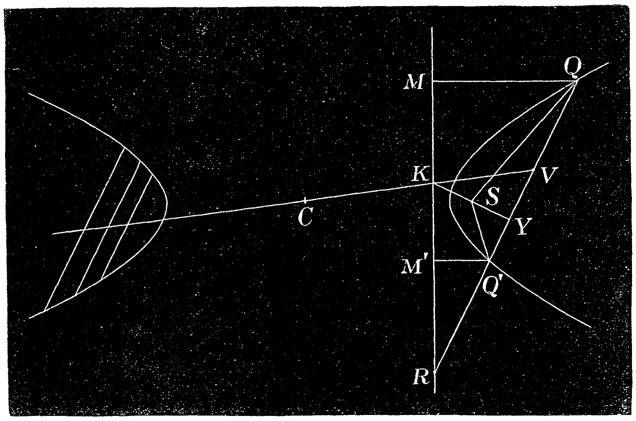
Ex. 10. A chord AP is divided in Q, so that $AQ : QP = CA^2 : CB^2$. Prove that the line through Q at right angles to QN is parallel to $A'P$.

PROPOSITION IX.

The locus of the middle points of any system of parallel chords of a hyperbola is a straight line passing through the centre.

Let QQ' be one of a system of parallel chords, and V its middle point.

Draw QM, $Q'M'$ perpendicular to the directrix; draw SY perpendicular to QQ' and produce YS to meet the directrix in K. Produce QQ' to meet the directrix in R, and join SQ, SQ'.



Then $SQ : QM = SQ' : Q'M'$

therefore $SQ : SQ' = QM : Q'M'$

$= QR : Q'R$;

therefore

$$SQ^2 - SQ'^2 : QR^2 - Q'R^2 = SQ^2 : QR^2.$$

But $SQ^2 - SQ'^2 = QY^2 - Q'Y^2$ [Euc. I. 47.

$= (QY + Q'Y)(QY - Q'Y)$

$= 2 . QQ' . YV$.

Similarly $QR^2 - Q'R^2 = 2 . QQ' . RV$;

therefore $YV : RV = SQ^2 : QR^2$.

Now, the ratio $SQ : QM$ is constant; also, the ratio $QM : QR$ is constant, since QQ' is drawn in a fixed direction. Therefore $SQ : QR$ is a constant ratio; therefore also $YV : RV$ is a constant ratio for all chords of the system. But as R always lies on a fixed straight line (the directrix), and Y on another fixed straight line (the focal perpendicular on the parallel chords), intersecting the former in K, V must also lie on a third fixed straight line, passing through the same point K.

Again, corresponding to a system of parallel chords in one branch of the hyperbola, there is in the other branch another system exactly similar thereto; and the middle points of the chords of both the systems must lie on VK, which therefore divides the two branches symmetrically, Hence, from the symmetry of the curve about the major and minor axes, and therefore about the centre, VK must pass through C.

Hence the diameter bisecting any system of parallel chords of a hyperbola is a straight line passing through the centre.

Ex. The diameter bisecting any system of parallel chords meets the directrix on the focal perpendicular to the chords.

* Proposition X.

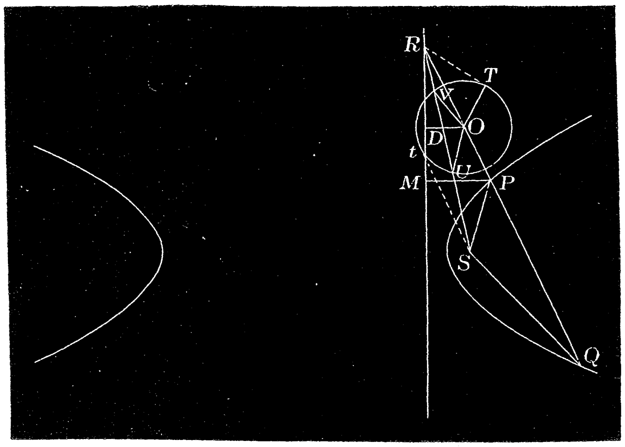
If any two parallel chords of a hyperbola be drawn through two fixed points, the ratio of the rectangles under their segments will be constant, whatever be the directions of the chords.

Let OPQ be a chord drawn through one of the fixed points O, outside the curve.

Produce QPO to meet the directrix in R, and join SR, SP, SQ. Draw OU, OV parallel to SP, SQ respectively; and draw OD, PM perpendicular to the directrix.

Then
$$\begin{aligned} RO:RP &= OU:PS \\ &= OD:PM, \end{aligned}$$

but $PS = e . PM$;
therefore $OU = e . OD$.
Similarly $OV = e . OD$.



Describe a circle with centre O and radius equal to $e . OD$, passing through U and V; and draw RT, St tangents to this circle.

Now, by similar triangles,

$$OP : OR = SU : RU,$$

and $$OQ : OR = SV : RV;$$

therefore $$OP . OQ : OR^2 = SU . SV : RU . RV$$
$$= St^2 : RT^2.$$ [Euc. III. 36.

Therefore $$OP . OQ : St^2 = OR^2 : RT^2.$$

Now for a given direction of the chord OPQ the ratio $OR : OD$ is constant, and, therefore, also the ratio $OR : OT$, since $OT = e . OD$. Therefore, also, the ratio $OR : RT$ is constant.

If, now, through another fixed point O' a parallel chord

$O'P'Q'$ be drawn, and similar constructions be made, we shall have $OP \,.\, OQ : St^2 = O'P' \,.\, O'Q' : St'^2$;
therefore $OP \,.\, OQ : O'P' \,.\, O'Q' = St^2 : St'^2$.
But since the points O and O' are fixed, the two circles are fixed in magnitude and position, and, therefore, St and St' are constants.

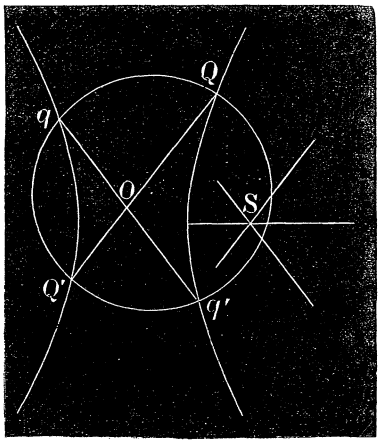
Therefore the ratio $OP \,.\, OQ : O'P' \,.\, O'Q'$ is constant.

Ex. 1. If a system of chords of a hyperbola be drawn through a fixed point, the rectangles contained by their segments are as the parallel focal chords, and also as the squares of the parallel semi-diameters where they exist. (Apply Prop. VI.)

Ex. 2. The ordinates to any diameter at equal distances from the centre are equal.

* Proposition XI.

If a circle intersect a hyperbola in four points, their common chords will be equally inclined, two and two, to the axis.



Let Q, Q' q, q' be the four points of intersection.

Then $QO \cdot OQ' = qO \cdot Oq'$. [Euc. III. 35.

Therefore the rectangles under the segments of the focal chords parallel to QQ' and qq' respectively are equal, [Prop. X.

and therefore the focal chords themselves are equal. [Prop. VI.

They are, therefore, equally inclined to the axis, from the symmetry of the figure. (See also Prop. I., Ex. 2.) Therefore, the chords QQ', qq' are equally inclined to the axis.

In like manner it may be shown that the chords Qq and $Q'q'$, as well as the chords Qq' and qQ', are equally inclined to the axis.

PROPERTIES OF TANGENTS.

The student should work out the following exercises as illustrating the method of deducing tangent properties from the corresponding chord-properties.

I. Deduce from Prop. X., Ex. 1 :—

1. The tangents to a hyperbola from an external point are proportional to the parallel semi-diameters where they exist, and are in the subduplicate ratio of the parallel focal chords.

2. If two parallel tangents OP, $O'P'$ be met by a third tangent at Q, in O and O', prove that

$$OP : O'P' = OQ : O'Q.$$

II. Deduce from Prop. XI. :—

1. PQ and PQ' are chords of a hyperbola equally inclined to the axis ; prove that the circle PQQ' touches the hyperbola at P.

2. If a circle touch a hyperbola at the points P and Q, show that PQ is parallel to one of the axes.

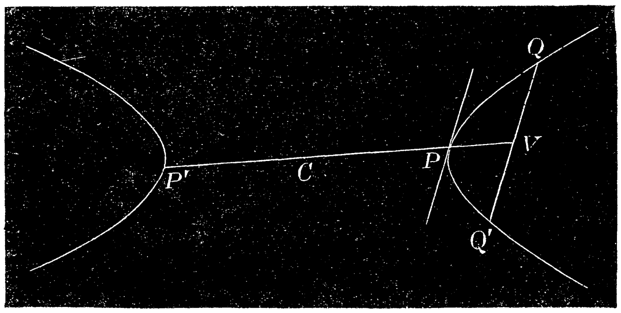
III. Deduce from Prop. VII., Ex. 1 :—

1. A tangent to one branch of a hyperbola cannot meet the other branch.

See also Prop. XII. and XIII.

PROPOSITION XII.

The tangent to a hyperbola at either end of a diameter is parallel to the system of chords bisected by the diameter.



Let $P'CPV$ be the diameter bisecting a system of chords parallel to QQ'. Let QQ' be made to move parallel to itself, so that Q may coincide with V. Since QV is always equal to $Q'V$, [Prop. X. it is clear that Q' will also coincide with V, and the chord in this its limiting position will be the tangent to the hyperbola at P.

Ex. 1. The tangent at the vertex is at right angles to the transverse axis.

Ex. 2. The line joining the points of contact of two parallel tangents is a diameter.

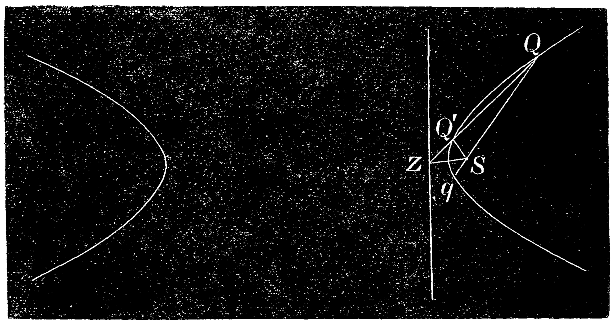
PROPOSITION XIII.

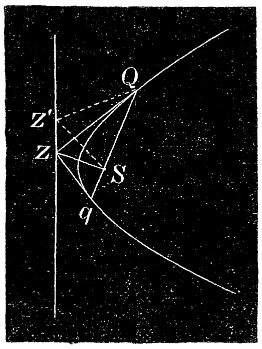
The portion of the tangent to a hyperbola at any point, intercepted between that point and the directrix, subtends a right angle at the focus, and conversely.

Also, tangents at the ends of a focal chord intersect on the directrix.

First, let any chord QQ' of the hyperbola intersect the directrix in Z; then SZ bisects the exterior angle $Q'Sq$. [Prop. VII.

Now, let the chord QQ' be made to turn about Q until the point Q' moves up to and coincides with Q, so that the chord becomes the tangent to the hyperbola at Q. In this limiting position of the chord QQ', since Q and Q' coincide, the angle QSQ' vanishes; therefore the angle $Q'Sq$ becomes equal to two right angles. But since SZ always bisects the angle $Q'Sq$, in this case the angle QSZ is a right angle.





Conversely, let QZ subtend a right angle at S, then it shall be the tangent to the hyperbola at Q. For if not and if possible let QZ' be the tangent at Q. Then the angle QSZ' is a right angle, which is impossible; therefore QZ is the tangent at Q.

Secondly, let QSq be a focal chord and QZ the tangent at Q.

Join ZS, Zq.

Then the angle QSZ being a right angle, the angle ZSq is also a right angle. Therefore qZ is the tangent to the hyperbola at q. Therefore the tangents at Q, q intersect on the directrix.

Ex. 1. Tangents at the extremities of the latus rectum intersect in X.

Ex. 2. To draw the tangent at a given point of a hyperbola.

Ex. 3. If QZ, qZ meet the latus rectum produced in D and d, then $SD = Sd$. (Cf. Chap. II., Prop. XV., Ex. 6.)

Proposition XIV.

If from a point O on the tangent at any point P of a hyperbola perpendiculars OU, OI be drawn to SP and the directrix respectively, then

$$SU = e \,.\, OI,$$

and conversely.

Join SZ and draw PM perpendicular to the directrix.

Because ZSP is a right angle, [Prop. XIII.

ZS is parallel to OU.

Therefore, by similar triangles,

$$SU : SP = ZO \cdot ZP$$
$$= OI : PM;$$

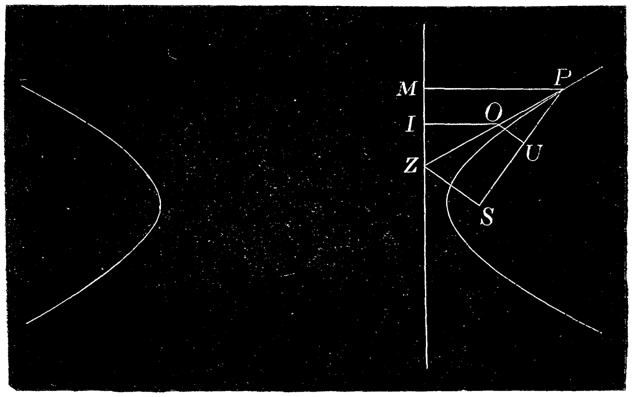
but $SP = e \,.\, PM$.

Therefore $SU = e \,.\, OI$.

Again, for the converse proposition, we have

$$SU = e \,.\, OI,$$

and
$$SP = e \,.\, PM$$



Therefore $SU : SP = OI : PM$
$= ZO : ZP.$

Therefore OU is parallel to ZS, [Euc. VI. 2.
and the angle PSZ is a right angle.
Therefore PZ is a tangent at P. [Prop. XIII.

Ex. If a perpendicular through O on the transverse axis meet the curve in Q and Q', then $SU = SQ$, and $OU^2 = OQ \,.\, OQ'$.

Proposition XV.

The tangent at any point of a hyperbola makes equal angles with the focal distances of the point.

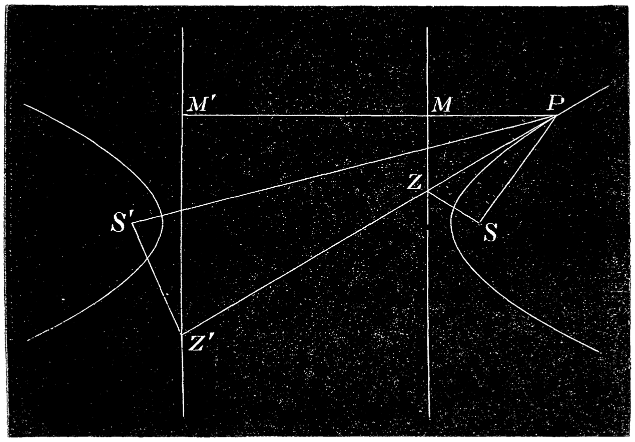
Let the tangent at P meet the directrices in Z and Z'.

Draw PMM' perpendicular to the directrices, meeting them in M and M' respectively. Join SP, SZ, $S'P$, $S'Z'$.

Then, in the two triangles PSZ and $PS'Z'$, the angles PSZ and $PS'Z'$ are equal, being right angles, [Prop. XIII.

and
$$SP:S'P=PM:PM'$$
$$=PZ:PZ',$$

and the angles PZS and $PZ'S'$ are both acute angles.



Therefore the triangles are similar. [Euc. VI. 7.

Therefore the angle $SPZ=$the angle $S'PZ'$.

Ex. 1. The tangent at the vertex is perpendicular to the axis.

Ex. 2. Given a focus, a tangent and its point of contact, find the locus of the other focus.

Ex. 3. If PCp be a diameter, and if Sp meet the tangent at P in T, $SP=ST$.

Ex. 4. If an ellipse and a hyperbola have the same foci, they intersect at right angles. (See Chap. I., Prop. XIV., Ex. 4.)

Such Conics are called *Confocal Conics*.

Ex. 5. If the tangent at P meet the axes in T, t, the angles PSt, STP are supplementary. [The circle round SPS' obviously passes through t.]

Ex. 6. If the diameter parallel to the tangent at P meet SP and $S'P$ in E and E', the circles about the triangles SCE, $S'CE'$ are equal.

Ex. 7. Tangents at the extremities of a focal chord PSQ meet in T. Prove that

$$2\angle PTQ - \angle PS'Q = 2 \text{ right angles.}$$

Ex. 8. Y, Y' are the feet of the focal perpendiculars on the tangent at P; if PN be the ordinate, the angles PNY, PNY' are supplementary. [$\angle PNY = \angle PSY = \angle PS'Y' = \pi - \angle PNY'$.]

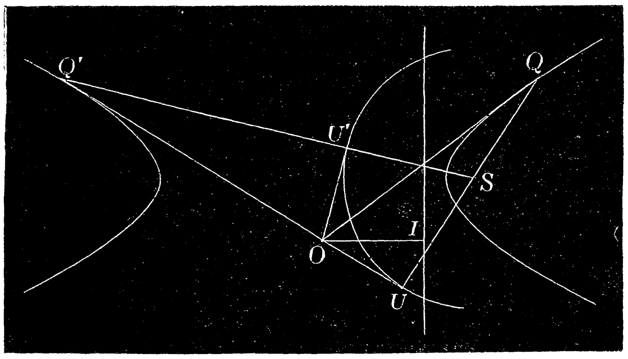
Ex. 9. A parabola and a hyperbola have a common focus S, and their axes in the same direction. A line SPQ cuts the curves in P and Q. If the tangents at P, Q meet in T, prove that $\angle PTQ = \frac{1}{2}\angle SS'Q$. (See Prop. I., Ex. 7.)

Ex. 10. P is a point on a hyperbola whose foci are S, S'; another hyperbola is described whose foci are S, P, and whose transverse axis $= SP - 2PS'$. Prove that the hyperbolas will meet at only one point, and that they will have the same tangent at that point. [Apply Prop. IV. If Q be a point of intersection, $QP = QS' + PS'$; Q, therefore, is the other extremity of the focal chord PS'.]

Ex. 11. A chord $PRVQ$ meets the directrices in R and V, P, Q being on different branches. Prove that PR and VQ subtend, each at the focus nearer to it, angles of which the difference is equal to the angle between the tangents at P and Q. (Apply Prop. VII.)

Proposition XVI.

To draw two tangents to a hyperbola from an external point.



Let O be the external point.

Draw OI perpendicular to the directrix, and with centre S and radius equal to $e . OI$ describe a circle. Draw OU, OU' tangents to this circle and let US and SU' produced meet the hyperbola in Q, Q'. Then OQ, OQ' shall be the tangents required.

For OU is at right angles to SU, [Euc. III. 18.
and $SU = e . OI$

Therefore OQ is the tangent to the hyperbola at Q. [Prop. XIV.

Similarly OQ' is the tangent at Q'.

Note.—If it had been necessary to produce *both* SU and SU' in the *same* direction, to meet the curve, the points of contact would have been on the same branch, instead of being on opposite branches, as in the figure.

Ex. 1. *Alternative Construction.*—With centre O and radius OS, describe a circle. With centre S' and radius equal to the transverse axis, describe another circle intersecting the former in M and M'. Join $S'M$ and $S'M'$, and produce them to meet the curve in Q and Q'. OQ, OQ' are the tangents required. (Cf. Chap. II., Prop. XVIII., Ex. 1.)

Ex. 2. Prove that only two tangents can be drawn to a hyperbola from an external point.

PROPOSITION XVII.

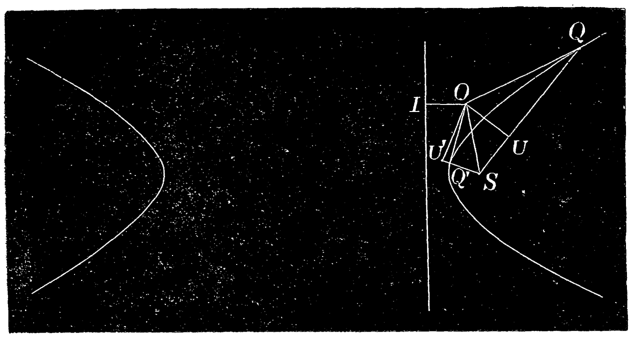
The two tangents that can be drawn to a hyperbola from an external point subtend equal or supplementary angles at the focus according as the points of contact are on the same or opposite branches of the curve.

First, let OQ, OQ' be the two tangents from O, Q and Q' being on the same branch of the curve.

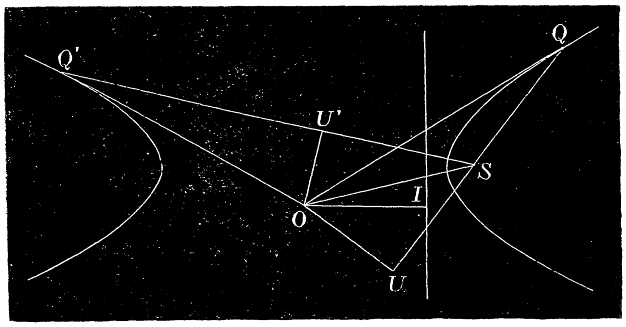
Join SO, SQ, SQ', and draw OI, OU, OU' perpendiculars upon the directrix, SQ, SQ' respectively.

Then $SU = e . OI = SU'$. [Prop. XIV.
Therefore $OU = OU'$. [Euc. I. 47.
Therefore the angles OSU and OSU' are equal, [Euc. I. 8.

and they are the angles which the tangents subtend at the focus.



Secondly, let Q and Q' be on opposite branches of the curve. Then it may be similarly proved that the angles



OSU and OSU' are equal; therefore the angles OSQ and OSQ' are supplementary.

Ex. 1. In Fig. 1 prove that OQ, OQ' subtend equal angles at S'.

Ex. 2. The portion of any tangent intercepted between the tangents at the vertices, subtends a right angle at either focus.

Ex. 3. Find the locus of the centre of the inscribed circle of the triangle SQS'. [The tangent at the vertex A.]

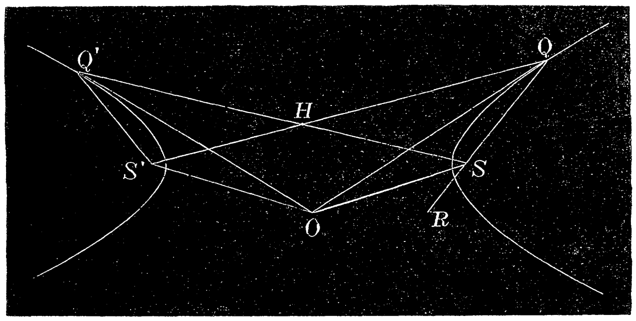
Ex. 4. Show that the chord of contact QQ' is divided harmonically by SO and the directrix.

Ex. 5. If PN be the ordinate of P, and PT the tangent, prove that
$$SP : ST = AN : AT.$$

Ex. 6. Two points P and Q are taken on the same branch of the curve and on the same side of the axis; prove that a circle can be drawn touching the four focal distances. [The centre is the point of intersection of the tangents at P and Q. Apply Prop. XV.]

* Proposition XVIII.

The two tangents that can be drawn to a hyperbola from an external point make equal or supplementary angles with the focal distances of the point according as the points of contact are on the opposite or same branches of the curve.



First, let OQ, OQ' be the two tangents from O, Q and Q' being on opposite branches of the curve.

Join SQ, SQ', SO, $S'Q'$, $S'Q$, $S'O$, and produce QS to R. Let H be the point of intersection of SQ' and $S'Q$.

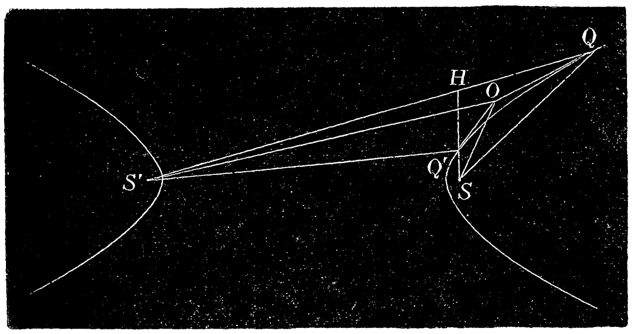
Then the angle SOQ

= the angle OSR − the angle OQS [Euc. I. 32.

= half the angle $Q'SR$ − half the angle SQS' [Props. XVII. and XV.

= half the angle SHQ.

Similarly the angle $S'OQ'$
$=$ half the angle $S'HQ'$.
Therefore the angle $SOQ =$ the angle $S'OQ'$. [Euc. I. 15.



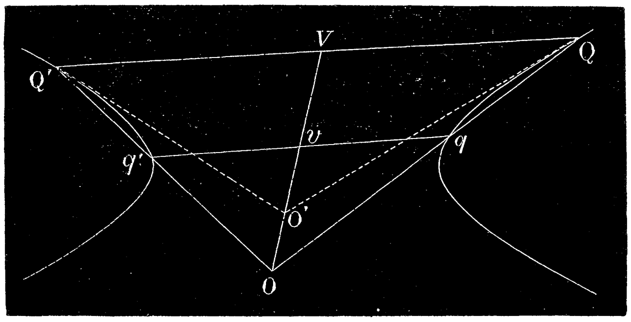
Secondly, let Q, Q' be on the same branch.
Then the angle SOQ
$=$ two right angles $-$ the angle $OSQ -$ the angle OQS [Euc. I. 32.
$=$ two right angles $-$ half the angle $QSQ' -$ half the angle SQS' [Prop. XVII. and XV.
$=$ two right angles $-$ half the angle SHS'. [Euc. I. 32.
Again, the angle $S'OQ'$
$=$ two right angles $-$ the angle $OQ'S' -$ the angle $OS'Q'$ [Euc. I. 32.
$=$ half the angle $SQ'S' -$ half the angle $QS'Q'$. [Props. XV. and XVII.
$=$ half the angle SHS'
Therefore, the angles SOQ and $S'OQ'$ are together equal to two right angles.

Ex. 1. Tangents are drawn from any point on a circle through the foci. Prove that the lines bisecting the angle between the tangents, or between one tangent and the other produced, all pass through a fixed point. [A point of intersection of the circle with the conjugate axis.]

Ex. 2. A hyperbola is described, touching the four sides (produced, if necessary) of a quadrilateral $ABCD$ which is inscribed in a circle. If one focus lies on the circle, the other also lies on it. [$\angle S'CD = \angle SCB = \angle SAB = \angle S'AD$.]

Proposition XIX.

The tangents at the extremities of any chord of a hyperbola intersect on the diameter which bisects the chord.



Let QQ' be the chord and qq' any other chord parallel to it.

Let Qq, $Q'q'$ produced meet in O. Bisect QQ' in V and let OV meet qq' in v.

Then $$QV : qv = OV : Ov$$
$$= Q'V : q'v;$$
but $$QV = Q'V,$$
therefore $$qv = q'v.$$
Thus OvV is the diameter bisecting the system of chords parallel to QQ'. [Prop. IX.

If now the chord qq' be made to move parallel to itself till it coincide with QQ', QqO and $Q'q'O$ will become the tangents to the curve at Q and Q' respectively. They thus meet on the diameter bisecting QQ'.

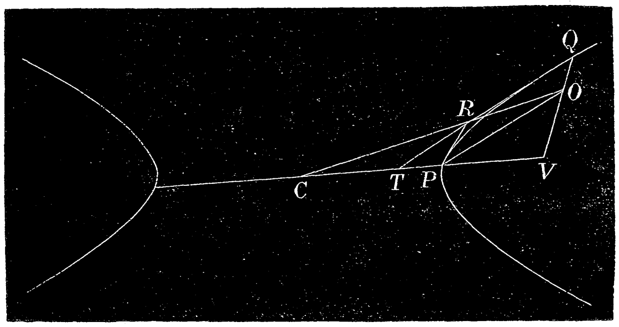
Ex. 1. Given a diameter of a hyperbola, to draw the system of chords bisected by it.

Ex. 2. If a circle passing through any point P on the curve, and having its centre on the normal at P, meets the curve again in Q and R, the tangents at Q and R intersect on a fixed straight line.

[The tangent at P and QR are equally inclined to the axis (see Prop. XI.); QR is, therefore, fixed in direction.]

PROPOSITION XX.

If the tangent at any point Q of a hyperbola meet any diameter CP in T and if QV be the ordinate to that diameter, $CV \, . \, CT = CP^2$.



Draw the tangent PR at P, meeting QT in R, and draw PO parallel to QT, meeting QV in O.

Then since $POQR$ is a parallelogram, [Prop. XII.
RO bisects PQ, and therefore passes through the centre C. [Prop. XIX.

By similar triangles

$$CV : CP = CO : CR = CP : CT,$$

therefore $$CV \, . \, CT = CP^2.$$

Note. When the diameter coincides with the transverse axis the result is stated thus:—

If the tangent at Q meets the transverse axis in T and QN be the perpendicular on the transverse axis.

$$CN \,.\, CT = CA^2.$$

From this it may be shown that

If the tangent at Q meets the conjugate axis, produced if necessary in t, and Qn be the perpendicular on the conjugate axis, $Cn \,.\, Ct = CB^2.$

$$\frac{QN \,.\, Ct}{CN \,.\, CT} = \frac{QN^2}{CN \,.\, NT} = \frac{QN^2}{CN^2 - CN \,.\, CT} = \frac{QN^2}{CN^2 - CA^2}.$$

$$\therefore \quad Cn \,.\, Ct = CB^2. \qquad \text{[Prop. VIII.}$$

These two results are important, and should be carefully noted by the student.

Ex. 1. If the tangent at Q meet the transverse axis in T and QN be the perpendicular on the transverse axis, show that

$$CN \,.\, NT = AN \,.\, NA'.$$

Ex. 2. In Ex. 1, if TD be drawn perpendicular to the axis to meet the circle described on AA' as diameter, then DN touches the circle.

Ex. 3. In Ex. 2, prove that

$$DN : QN = CA : CB.$$

Also if DA be produced to meet PN in K,

$$QN : NK = CB : CA.$$

(Apply Prop. VIII., and see Ex. 1.)

Ex. 4. Any diameter is cut harmonically by a tangent and the ordinate of the point of contact of the tangent with respect to the diameter.

Ex. 5. Any tangent is cut harmonically by any two parallel tangents and the diameter through their points of contact. (Ex. 4.)

Ex. 6. If PN be the ordinate of a point P, and NQ be drawn parallel to AP to meet CP in Q, AQ shall be parallel to the tangent at P.

Ex. 7. If the tangent at P intersect the tangents at the vertices and the transverse axis in R, r and T, show that

(i) $AT \,.\, A'T = CT \,.\, TN.$

(ii) $AR \,.\, A'r = CB^2.$

Ex. 8. P is any point on a hyperbola. Prove that the locus of the centre (Q) of the circle touching SP, $S'P$ produced, and the transverse axis, is a hyperbola.

[Let QM be the ordinate of Q; then, if the tangents at A and P meet in F, QSF is a right angle, and

$$\frac{QM}{MS}=\frac{SA}{AF}, \quad \frac{QM}{MS'}=\frac{S'A'}{A'F'};$$

$$\therefore \quad QM^2 : MS \,.\, MS' = SA^2 : CB^2. \qquad \text{Ex. 7.}$$

Then apply Prop. VIII., Ex. 6.]

Ex. 9. The tangent at P bisects any straight line perpendicular to AA', and terminated by AP and $A'P$.

[Let the tangent at P, AP, $A'P$ meet the conjugate axis in t, E, E' respectively. Then

$$\frac{CE-CE'}{PN}=\frac{CA \,.\, A'N - CA' \,.\, AN}{AN \,.\, A'N}=\frac{2CA^2}{AN \,.\, A'N}=\frac{2Ct}{PN}.$$

[Prop. VIII.

$\therefore$ $CE-CE'=2Ct$, or t bisects EE'.]

Ex. 10. An ellipse and a hyperbola are described, so that the foci of each are at the extremities of the transverse axis of the other; prove that the tangents at their points of intersection meet the conjugate axis in points equidistant from the centre. [The conjugate axes of the two curves are equal in length.]

PROPOSITION XXI.

The locus of the foot of the perpendicular drawn from either focus upon any tangent to a hyperbola is the circle described on the transverse axis as diameter; and the rectangle under the focal perpendiculars on the tangent is equal to the square of the semi-conjugate axis.

$$(SY \,.\, S'Y' = CB^2.)$$

Let SY, $S'Y'$ be the focal perpendiculars upon the tangent at any point P.

Join SP, $S'P$, and produce SY to meet $S'P$ in R. Join CY.

Then in the triangles SPY, RPY, the angle SPY = the angle RPY, [Prop. XV.

the angles SYP, RYP are equal, being right angles, and YP is common.

Therefore $SP = PR$, $SY = YR$. [Euc. I. 26.

Also $SC = CS'$;

therefore CY is parallel to $S'P$. [Euc. VI. 2.

Therefore $CY = \frac{1}{2}S'R$ [Euc. VI. 4.

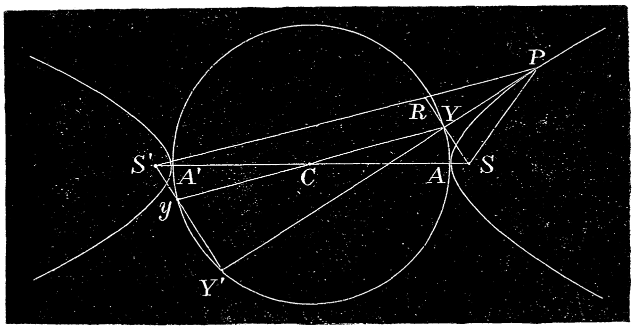
$= \frac{1}{2}(S'P - PR)$

$= \frac{1}{2}(S'P - SP)$

$= \frac{1}{2}AA'$ [Prop. IV.

$= CA$;

therefore the locus of Y is the circle described on the transverse axis as diameter.



Similarly it may be shown that the locus of Y' is the same circle.

Again, produce YC to meet $S'Y'$ in y. Then y will be on the circle.

For, since $CS = CS'$, and SY is parallel to $S'Y'$, the triangles SCY, $S'Cy$ are equal. [Euc. I. 26.

Therefore $Cy = CY = CA$, showing that y is on the circle.

Also $SY = S'y$,

therefore $SY . S'Y' = S'y . S'Y' = S'A' . S'A$ [Euc. III. 35.

$= SA . SA' = CB^2$. [Def.

Ex. 1. If CE drawn parallel to the tangent at P meet $S'P$ in E, then $PE=CA$.

Ex. 2. From a point on the circle on AA' as diameter lines are drawn touching the curve in P, P'. Prove that SP', $S'P$ are parallel. [Each is parallel to CY.]

Ex. 3. If through any point Y on the circle on AA' as diameter YP be drawn at right angles to SY, YP will be a tangent to the hyperbola.

Ex. 4. If the vertex of a right angle moves on a fixed circle, and one leg passes through a fixed point *outside* the circle, the other leg will always touch a hyperbola.

Ex. 5. Given a focus, a tangent, and a point on a hyperbola, find the locus of the other focus. [An arc of a fixed hyperbola of which the foci are the given point and the image of the focus in the tangent.]

Ex. 6. Given a focus, a tangent, and the transverse axis, find the locus of the other focus. [A circle ; centre R, radius $=AA'$.]

Ex. 7. If PN be the ordinate of P, the points Y, Y', N, C lie on a circle.

Ex. 8. The right lines joining each focus to the foot of the perpendicular from the other focus on the tangent meet on the normal and bisect it.

Ex. 9. Alternative Construction for Prop. XVI.

Let O be the external point. On OS as diameter describe a circle, cutting the circle on AA' as diameter in Y and Y'. Then OY and OY' produced will be the tangents required.

Ex. 10. If tangents be drawn from P to a circle described with S' as centre and radius equal to CB, the chord of contact will touch the circle described on AA' as diameter. [The line through y perpendicular to $S'P$ will be the chord of contact.]

Ex. 11. If the tangent at P cuts the transverse axis in T, prove that $AT \cdot A'T = YT \cdot Y'T$.

Ex. 12. Find the position of P when the area of the triangle YCY' is the greatest possible.

[$CY=CY'=CA$; therefore YCY' must be a right angle. If the tangent at P meets CB in t, $PN \cdot Ct=CB^2$. (Prop. XX.) Also the triangles CYS, $CY't$ are equal ; therefore $PN \cdot CS=CB^2$.]

Ex. 13. If SY, SZ be perpendiculars on two tangents which meet in O, OZ is perpendicular to $S'O$. [$S'O$ is parallel to the bisector of YCZ. Apply Prop. XVII.]

Ex. 14. An ellipse and a hyperbola are confocal ; if a tangent to the one intersects at right angles a tangent to the other, the locus of the point of intersection is a circle.

Let SY, $S'Y'$ be the focal perpendiculars upon the tangent to the ellipse, and SZ, $S'Z'$ those upon the tangent to the hyperbola ; let the tangents meet at O ; let a, b be the semi-axes of the ellipse, and α, β those of the hyperbola. Then if CV be perpendicular to YOY',

$$OY . OY' = Y'V^2 - OV^2$$

and $$CO^2 + OY . OY' = CY'^2 = CA^2 ;$$

$\therefore$ $$CO^2 + SZ . S'Z' = a^2$$

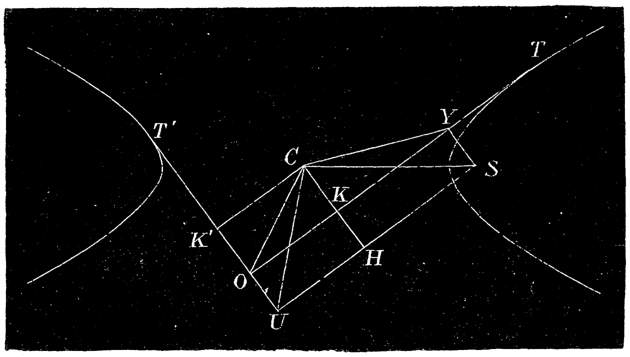
or $$CO^2 = a^2 - \beta^2.$$

See also Prop. IV., Ex. 14, 15.

Ex. 15. If an ellipse and a hyperbola are confocal, the difference of the squares of the central distances of parallel tangents is constant ($= b^2 + \beta^2$. Ex. 14.)

* Proposition XXII.

The locus of the intersection of tangents to a hyperbola which cut at right angles is a circle.



Let the tangents OT, OT' cut at right angles at O.

Draw SY, CK perpendicular to OT, and SU, CK' perpendicular to OT'. Join CY, CU, CO, and produce CK to meet SU in H.

Now Y and U are on the circle on AA' as diameter; [Prop. XXI.

therefore $CY = CU = CA$.

Now $CO^2 = CK^2 + CK'^2$, [Euc. I. 47.

and $CY^2 = CK^2 + YK^2$;

therefore $CA^2 = CK^2 + SH^2$.

Also $CU^2 = CK'^2 + UK'^2$,

therefore $CA^2 = CK'^2 + HC^2$;

therefore
$$\begin{aligned} 2CA^2 &= CK^2 + CK'^2 + SH^2 + HC^2 \\ &= CO^2 + CS^2. \end{aligned}$$ [Euc. I. 47.

But $CS^2 = CA^2 + CB^2$; [Def.

therefore $CO^2 = CA^2 - CB^2$.

Hence the locus of O is a circle described with centre C.

Note.—This circle is called the *director circle* of the hyperbola. In the case when CB is greater than CA, $CA^2 - CB^2$ is negative, and, therefore, the locus does not exist, that is, when CB is greater than CA the hyperbola has no tangents cutting at right angles.

Ex. Four tangents to a hyperbola form a rectangle; if one side UV of the rectangle intersect a directrix in F, and S be the corresponding focus, the triangles FSU, FVS are similar.

[$SF^2 = CF^2 - CX^2 + SX^2$
$= CF^2 + CS^2 - 2CS \, . \, CX = CF^2 - CA^2 + CB^2$
$=$ square of tangent from F to the director circle
$= FU \, . \, FV$.]

PROPERTIES OF NORMALS.

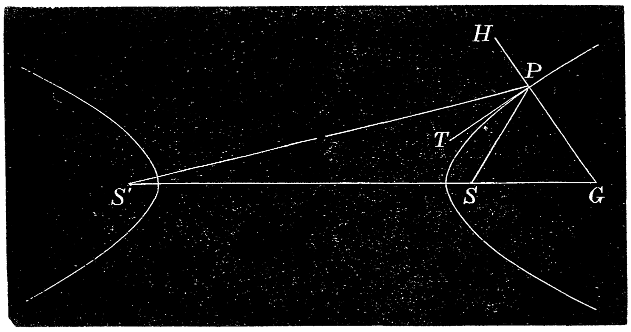
Proposition XXIII.

The normal at any point of a hyperbola makes equal angles with the focal distances of the point.

Let the normal HPG at the point P meet the axis in G.

Let PT be the tangent at P. Then

the angle $SPT =$ the angle $S'PT$. [Prop. XV.



But the angles TPG and TPH are equal, being right angles; [Def.

therefore the angle $SPG =$ the angle $S'PH$.

Ex. 1. If the tangent and normal at P meet the conjugate axis in t and g, P, t, g, S, S' lie on the same circle.

Ex. 2. If a circle through the foci meet two confocal hyperbolas in P and Q, the angle between the tangents at P and Q is equal to PSQ.

Ex. 3. The tangent at P meets the conjugate axis in t, and tQ is perpendicular to SP. Prove that SQ is of constant length.

[If SY is perpendicular to Ct, $CY = CA$. Prop. XXI. Also Q, S, C, t lie on a circle. $\therefore \angle tQC = \angle tSS' = \angle tPS' = \angle tPS$. $\therefore CQ \parallel SY$, and $SQ = CY = CA$.]

Ex. 4. If from g a perpendicular gK be drawn on SP, show that $PK = CA$. (Cf. Chap. II., Prop. XXVI., Ex. 3.)

Ex. 5. Prove that $SP \, . \, S'P = PG \, . \, Pg$. (Cf. Chap. II., Prop. XXVI., Ex. 4.)

* PROPOSITION XXIV.

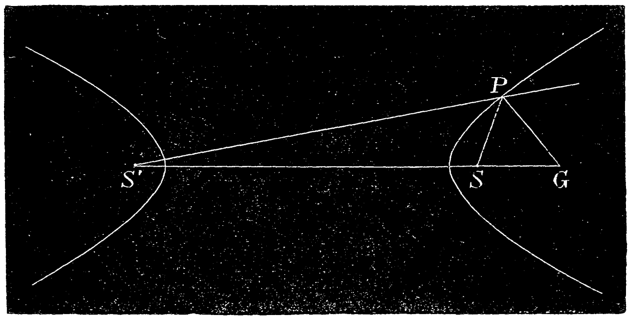
If the normal at any point P of a hyperbola meet the transverse axis in G,

$$SG = e \, . \, SP.$$

Join $S'P$.

Then, since PG bisects the exterior angle between SP and $S'P$,

$$S'G : SG = S'P : SP; \qquad \text{[Euc. VI. A.}$$



therefore $$S'G - SG : SG = S'P - SP : SP,$$

or $$SG : SP = S'G - SG : S'P - SP.$$

But $$S'G - SG = SS' = e . AA', \qquad \text{[Prop. III.}$$

and $$S'P - SP = AA'; \qquad \text{[Prop. IV.}$$

therefore $$SG = e . SP.$$

Ex. 1. The projection of the normal upon the focal distance of any point is equal to the semi-latus rectum. (Cf. Chap. II., Prop. XXVII., Ex. 4.)

Ex. 2. A circle passing through a focus, and having its centre on the transverse axis, touches the curve; prove that the focal distance of the point of contact is equal to the latus rectum.

Ex. 3. Draw the normal at any point without drawing the tangent.

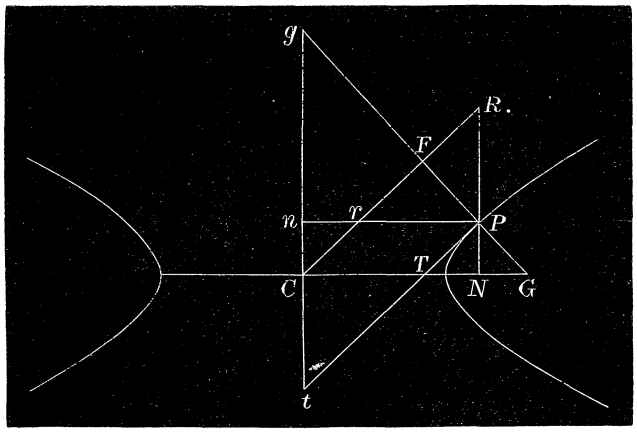
* PROPOSITION XXV.

The normal at any point of a hyperbola terminated by either axis varies inversely as the central perpendicular upon the tangent.

$$(PG . PF = CB^2; \; Pg . PF = CA^2.)$$

Let the normal at P meet the transverse and conjugate axis in G and g respectively, and let the tangent at P meet them in T and t respectively.

Draw PN, Pn perpendicular to the transverse and conjugate axis, and let a straight line through the centre, drawn parallel to the tangent at P, meet NP, GP produced and Pn in R, F, and r respectively.



Then, since the angles at N and F are right angles, G, N, F, R lie on a circle.

Therefore $PG \, . \, PF = PN \, . \, PR$ [Euc. III. 35.

$= Cn \, . \, Ct$ [Euc. I. 34.

$= CB^2$. [Prop. XX., Note.

Again, since the angles at n and F are right angles, g, F, r, n lie on a circle.

Therefore $Pg \, . \, PF = Pn \, . \, Pr$ [Euc. III. 36.

$= CN \, . \, CT$ [Euc. I. 34.

$= CA^2$. [Prop. XX., Note.

Therefore both PG and Pg vary inversely as PF, which is equal to the central perpendicular upon the tangent at P.

Ex. In Prop. XXIII., Ex. 1, prove that

$$Gg = e \, . \, Sg$$

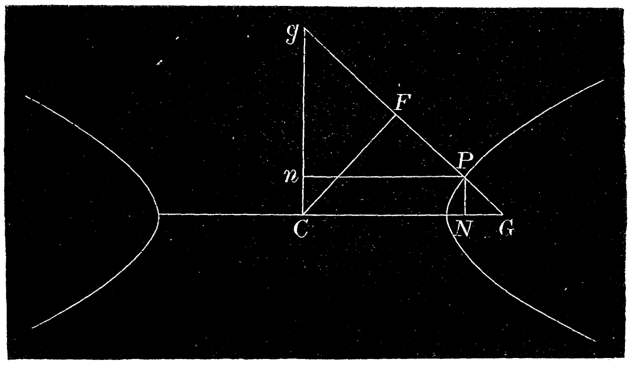
Apply Prop. III., Ex. 2.

* PROPOSITION XXVI.

If the normal at any point P of a hyperbola meet the transverse axis in G, and PN be the ordinate to that axis,

$$\text{(i)} \quad GN : CN = CB^2 : CA^2.$$

$$\text{(ii)} \quad CG = e^2 \, . \, CN$$



Let the normal meet the conjugate axis in g. Draw Pn perpendicular to the conjugate axis, and CF parallel to the tangent at P.

Then, because the triangles PNG and Png are similar,

$$GN : CN = PG : Pg \qquad \text{[Euc. VI. 2.}$$
$$= PG \, . \, PF : Pg \, . \, PF$$
$$= CB^2 : CA^2; \qquad \text{[Prop. XXV.}$$

therefore $\quad CN + GN : CN = CA^2 + CB^2 : CA^2,$

or $\quad CG : CN = CS^2 : CA^2. \qquad$ [Def.

But $CS = e \, . \, CA$; [Prop. III.
therefore $CG = e^2 \, . \, CN$.

Ex. 1. Prove that

$$CG \, . \, Cn : Cg \, . \, CN = CB^2 : CA^2.$$

Ex. 2. Show that

$$Sn : Cn = CA^2 : CB^2.$$

Ex. 3. If the tangent and normal at P meet the axis in T and G, prove that

(i) $NG \, . \, CT = CB^2$. (ii) $CG \, . \, CT = CS^2$.

[Apply Prop. XX.]

Ex. 4. Find the locus of the points of contact of tangents to a series of confocal hyperbolas from a fixed point on the axis.

[From Ex. 3 (ii), G the foot of the normal is fixed; hence P lies on the circle of which TG is diameter.]

PROPERTIES OF ASYMPTOTES.

Def. When a curve continually approaches to a fixed straight line without ever actually meeting it, but so that its distance from it, measured along any straight line, becomes ultimately less than any finite length, the fixed straight line is called an *asymptote* to the curve.

Proposition XXVII.

The diagonals of the rectangle formed by perpendiculars to the axes of a hyperbola, drawn through their extremities, are asymptotes to the curve.

Let CR, CR' be the diagonals of the rectangle formed by perpendiculars through the extremities A, A', B, B' of the axes of the hyperbola. Through any point N on the transverse axis draw $pPNP'p'$ perpendicular to it, meeting the curve in P and P', and CR, CR' in p, p' respectively.

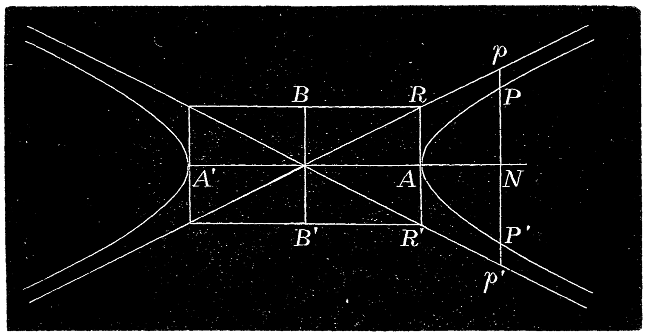
Now $PN^2 : AN \, . \, A'N = CB^2 : CA^2$, [Prop. VIII.
or $PN^2 : CN^2 - CA^2 = CB^2 : CA^2$. [Euc. II. 6.

Again $pN^2 : CN^2 = AR^2 : CA^2$

$= CB^2 : CA^2$;

therefore $pN^2 - PN^2 : CA^2 = CB^2 : CA^2$,

or $pN^2 - PN^2 = CB^2$.



But since pp' is bisected in N,

$pN^2 - PN^2 = pP \,.\, p'P$. [Euc. II. 5.

Therefore $pP \,.\, p'P = CB^2$.

Now $p'P = NP + Np'$,

and NP^2 varies as $AN \,.\, A'N$, [Prop. VIII.

and Np' varies as CN.

Hence, as N moves along $A'A$ produced, both NP and $N'p$, and therefore also Pp', continually increase. But the product $pP \,.\, p'P$, of which one factor $p'P$ continually increases, is constant; therefore $p'P$ continually diminishes, and becomes ultimately less than any finite length, however small. CR, therefore, is an asymptote to the hyperbola. Similarly, CR' is another asymptote.

Ex. 1. The lines joining the extremities of the axes are bisected by one asymptote and parallel to the other.

Ex. 2. Any line parallel to an asymptote cannot meet the curve in more than one point.

Ex. 3. Prove that the angle between the asymptotes of the hyperbola in Prop. I., Ex. 10, is double the exterior angle between the tangents.

Ex. 4. The circle on AA' as diameter cuts the directrices in the same points as the asymptotes.

Ex. 5. If the directrix meets CR in F, prove that (i) $CF=AC$; (ii) CFS is a right angle.

Ex. 6. Given one asymptote, the direction of the other, and the position of one focus, find the vertices.

Ex. 7. If CR meets the directrix in F, AF is parallel to SR.

Ex. 8. Given the asymptotes and a focus to find the directrix. [Apply Ex. 5 (ii).]

Ex. 9. Given the centre, an asymptote, and a directrix, to find the focus. [Apply Ex. 5 (ii).]

Ex. 10. Given an asymptote, the directrix, and a point on the hyperbola, to construct the curve. (Ex. 5.)

Ex. 11. The straight line drawn from the focus to the directrix, parallel to an asymptote, is equal to the semi-latus rectum, and is bisected by the curve. (Cf. Ex. 13.)

Ex. 12. The perpendicular from the focus on either asymptote is equal to the semi-conjugate axis.

Ex. 13. The focal distance of any point on the curve is equal to the length of the line drawn from the point parallel to an asymptote to meet the directrix. (Cf. Ex. 11.)

Ex. 14. Given the eccentricity of a hyperbola, find the angle (θ) between the asymptotes. $\left(\sec\frac{\theta}{2}=e.\right)$

Ex. 15. Prove that the tangents to a hyperbola from C coincide with the asymptotes.

Apply Prop. XVI., Ex. 1, observing that the tangents are lines bisecting SM, SM' at right angles.

The asymptotes may thus be regarded as tangents to the hyperbola whose points of contact are at infinity.

Ex. 16. If the tangent at P meets an asymptote in T, prove that ST will bisect the angle between PS and the line through S parallel to the asymptote. (Apply Ex. 15 and Prop. XVII.)

Ex. 17. If the tangent at P meets an asymptote in T, prove that $\angle STP=\angle S'TC=\angle PS'T$. (Ex. 15.)

Ex. 18. If a tangent meet the asymptotes in L and M, the angle subtended by LM at the farther focus is half the angle between the asymptotes.

[Apply Ex. 16 and Prop. XVIII. If $S'L'$, $S'M'$ be drawn parallel to the asymptotes, LS', MS' bisect the angles $PS'L'$, $PS'M'$.]

Ex. 19. Given an asymptote, the focus, and a point on the hyperbola to construct the curve.

[The feet of the focal perpendiculars on the asymptote and the tangent at the point (Ex. 16) will lie on the circle described on AA' as diameter (Ex. 15 and Prop. XXI.), whence the centre is determined ; the directrix is found at once by Ex. 5.]

Ex. 20. The tangent and normal at any point meet the asymptotes and the axes respectively in four points lying on a circle, which passes through the centre of the hyperbola, and of which the radius varies inversely as the central perpendicular on the tangent.

Ex. 21. The radius of the circle which touches a hyperbola and its asymptotes is equal to the part of the latus rectum intercepted between the curve and an asymptote. (Apply Prop. V.)

Ex. 22. A parabola P and a hyperbola H have a common focus, and the asymptotes of H are tangents to P. Prove that the tangent at the vertex of P is a directrix of H, and that the tangent to P at its intersection with X passes through the farther vertex of H.

[The line joining the feet of the focal perpendiculars upon the asymptotes is the tangent at the vertex of P (Chap. I., Prop. XXIII.), and the directrix of H (Ex. 5). If P be a common point, and PM be perpendicular to the directrix of H, we have $SP : PM = SC : CA$, and $SP = PM + SX$. $\therefore\ SP : SX = CS : AS$. $\therefore\ SP \,.\, AS = SX \,.\, CS = CB^2 = SA \,.\, SA'$. $\therefore\ SP = SA'$ and $A'P$ touches the parabola at P. (Chap. I., Prop. XIV.).]

Ex. 23. If an ellipse and a confocal hyperbola intersect in P, an asymptote passes through the point on the auxiliary circle corresponding to P. (Apply Prop. IV., Ex. 13.)

Proposition XXVIII.

If through any point on a hyperbola a straight line parallel to either axis be drawn meeting the asymptotes, the rectangle under its segments is equal to the square of the semi-axis to which it is parallel.

First case.

Through any point P on the hyperbola draw Ppp' parallel to the transverse axis, meeting the asymptotes in p and p' and the conjugate axis in n.

Then, since pp' is bisected at n,

$$Pp \,.\, Pp' = Pn^2 - pn^2. \qquad \text{[Euc. II. 6.}$$

Now $PN^2 : AN \,.\, A'N = CB^2 : CA^2$, [Prop. VIII.

therefore $PN^2 : CN^2 - CA^2 = CB^2 : CA^2$; [Euc. II. 6.

or $Cn^2 : Pn^2 - CA^2 = CB^2 : CA^2$,

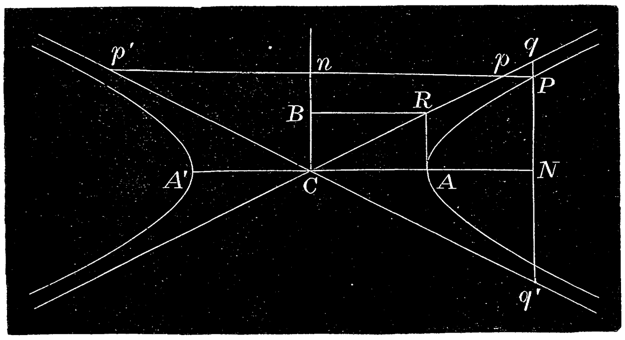
but $Cn^2 : pn^2 = CB^2 : BR^2$

$= CB^2 : CA^2$,

therefore $PN^2 - CA^2 = pn^2$,

or $Pn^2 - pn^2 = CA^2$,

therefore $Pp \,.\, Pp' = CA^2$.



Second case.

Through P draw qPq' parallel to the conjugate axis, meeting the asymptotes in q, q'.

Then, as before,

$PN^2 : CN^2 - CA^2 = CB^2 : CA^2$,

or $PN^2 + CB^2 : CN^2 = CB^2 : CA^2$,

or $PN^2 + CB^2 : Pn^2 = CB^2 : CA^2$;

but $qN^2 : Pn^2 = qN^2 : CN^2$

$= AR^2 : CA^2$

$= CB^2 : CA^2$;

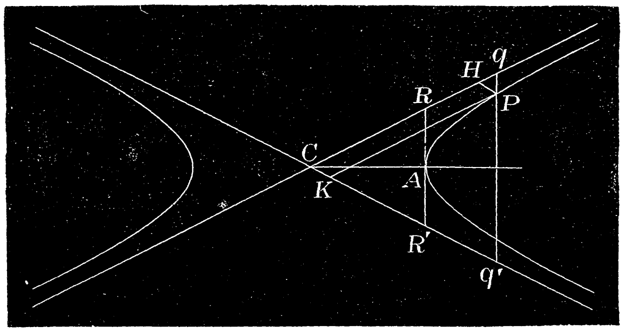
therefore $qN^2 = PN^2 + CB^2$,

or $qN^2 - PN^2 = CB^2$,

or $Pq \,.\, Pq' = CB^2$. [Euc. II. 6.

PROPOSITION XXIX.

If through any point on a hyperbola lines be drawn parallel to the asymptotes, the rectangle under the segments intercepted between the point and the asymptotes is constant.



Through any point P on the hyperbola draw PH, PK parallel to the asymptotes, meeting them in H, K. Draw RAR' and qPq' perpendicular to CA.

Then, by similar triangles,

$$PH : Pq = CR' : RR',$$

and $$PK : Pq' = CR : RR',$$

therefore $$PH . PK : Pq . Pq' = CR' . CR : RR'^2,$$

or $$PH . PK : CB^2 = CR^2 : 4RA^2. \quad \text{[Prop. XXVIII.}$$

$$= CA^2 + CB^2 : 4CB^2$$

$$= CS^2 : 4CB^2. \quad \text{[Def.}$$

or $$PH . PK = \tfrac{1}{4} CS^2.$$

Ex. 1. Find the locus of the point of intersection of the medians of the triangle formed by a tangent with the asymptotes. [A hyperbola having the same asymptotes.]

Ex. 2. P, Q are points on a hyperbola. PL, QM are drawn parallel to each other to meet one asymptote; PR, QN are drawn also parallel to each other to meet the other asymptote. Prove that $PL . PR = QM . QN$.

Ex. 3. If through P, P' on a hyperbola lines are drawn parallel to the asymptotes, forming a parallelogram, one of its diagonals will pass through the centre.

Ex. 4. If P be the middle point of a line which moves so as to form with two intersecting lines a triangle of constant area, the locus of P is a hyperbola.

Ex. 5. If through any point of a hyperbola, lines be drawn parallel to the asymptotes meeting any semi-diameter CQ in P and R, then $CP . CR = CQ^2$.

Ex. 6. A series of hyperbolas having the same asymptotes is cut by a fixed straight line parallel to one of the asymptotes, and through the points of intersection lines are drawn parallel to the other, and equal to either axis of the corresponding hyperbola; prove that the locus of their extremities is a parabola.

Ex. 7. Given the asymptotes and a point on the curve, to construct it. (Apply Prop. XXVII., Ex. 5.)

Ex. 8. If a line through the centre meets PH, PK in U, V, and the parallelogram $PUQV$ be completed, prove that Q is on the curve.

[If QU, VQ meet the asymptotes in U', V', since the parallelograms HK, $U'V'$ are equal, $PH . PK = QU' . QV'$.]

Ex. 9. The ordinate NP at any point of an ellipse is produced to Q, such that NQ is equal to the subtangent at P. Prove that the locus of Q is a hyperbola.

[If P is on the quadrant AB, the asymptotes are CB and the bisector of the angle ACB'.]

Ex. 10. If a straight line passing through a fixed point C, meets two fixed lines OA, OB in A, B, and if P be taken on AB such that $CP^2 = CA . CB$, find the locus of P.

[Through C draw CD, CE parallel to OA, OB, to meet them. Through P draw lines parallel to OA, OB meeting CE in K, and DC in H. Then $OD . OE = PH . PK$. The locus of P is, therefore, a hyperbola of which the asymptotes are CH, CK.]

Def. Two hyperbolas are said to be *conjugate* when the transverse axis of each coincides with the conjugate axis of the other.

Thus, a hyperbola which has CB and CA for transverse and conjugate axes respectively, is called the *Conjugate hyperbola*, with reference to the one we have been dealing with.

The conjugate hyperbola has the same asymptotes as the original one, since they are the diagonals of the same rectangle. It is evident that a pair of conjugate hyperbolas lie on opposite sides of their common asymptotes.

It has already been pointed out that the *two* branches of a hyperbola together constitute *one* complete curve; but it must not, by analogy, be supposed that a pair of conjugate hyperbolas together constitutes *one entire* curve. They are a pair of totally distinct hyperbolas, although one is of use in deducing some properties of the other.

Ex. 1. Tangents TP, TQ are drawn to a hyperbola from any point T on one of the branches of the conjugate. Prove that PQ touches the other branch of the conjugate.

[CT bisects PQ in V, Prop. XIX.; and $CT.CV=CT^2$. Prop. XX.]

Ex. 2. An ordinate NP meets the conjugate hyperbola in Q; prove that the normals at P and Q meet on the transverse axis.

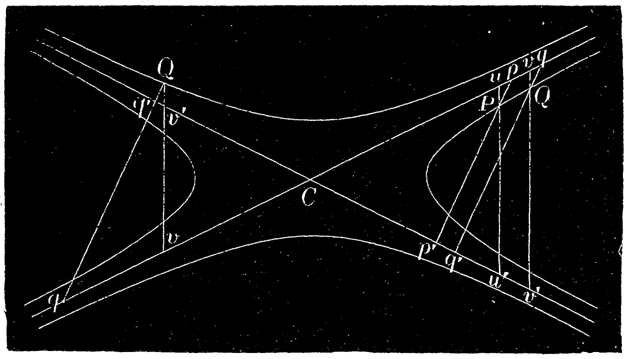
[If the normal at Q meets the axes in G and G',

$$\frac{QG'}{QG}=\frac{CA^2}{CB^2}=\frac{CN}{NG}.$$

Apply Props. XXV., XXVI.]

PROPOSITION XXX.

If through any point on a hyperbola or its conjugate a straight line be drawn in a given direction to meet the asymptotes, the rectangle under its segments is constant.



Let P be the point on the given hyperbola and Q a point either on the same hyperbola or its conjugate.

Draw pPp' and qQq' in the given direction, meeting the asymptotes in p, p' and q, q' respectively. Through P, Q draw uPu', vQv' parallel to the conjugate axis, meeting the asymptotes in u, u' and v, v' respectively.

Now, by similar triangles,

$$Pp : Qq = Pu : Qv,$$

and $$Pp' : Qq' = Pu' : Qv',$$

therefore $Pp \,.\, Pp' : Qq \,.\, Qq' = Pu \,.\, Pu' : Qv \,.\, Qv'$;

but $$Pu \,.\, Pu' = CB^2 = Qv \,.\, Qv', \quad \text{[Prop. XXVIII.}$$

therefore $$Pp \,.\, Pp' = Qq \,.\, Qq'.$$

Ex. 1. Prove that

$$Pp \,.\, Pp' = Qq \,.\, Qq' = CD^2,$$

where CD is the parallel semi-diameter terminated by the curve or its conjugate.

Ex. 2. An ordinate QV of any diameter CP is produced to meet the asymptote in R, and the conjugate hyperbola in Q'. Prove that $$QV^2 + Q'V^2 = 2RV^2.$$
Prove also that the tangents at Q, Q' meet CP in points equidistant from C. [$Q'V^2 - RV^2 = CD^2$. For the second part, apply Prop. XX.]

Proposition XXXI.

If any line cut a hyperbola the segments intercepted between the curve and its asymptotes are equal, and the portion of any tangent intercepted between the asymptotes is bisected at the point of contact.

Let any line meet the curve and its asymptotes in Q, Q' and q, q' respectively.

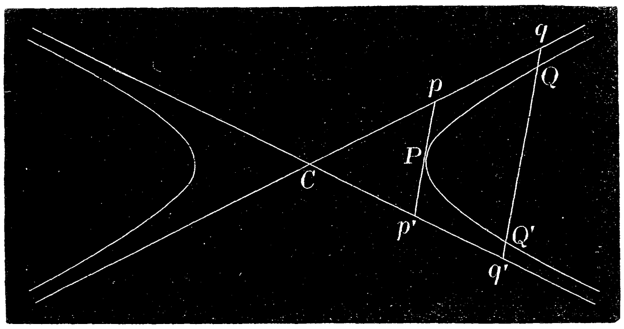
Now $$Qq \,.\, Qq' = Q'q \,.\, Q'q'. \quad \text{[Prop. XXX.}$$

or $$Qq \,.\, QQ' + Qq \,.\, Q'q' = Q'q' \,.\, QQ' + Qq \,.\, Q'q', \quad \text{[Euc. II. 1}$$

or $$Qq \,.\, QQ' = Q'q' \,.\, QQ',$$

therefore $$Qq = Q'q'.$$

If now QQ' be made to move parallel to itself until the points Q, Q' coincide at a point P on the curve it becomes the tangent to the curve at P and $Pp = Pp'$



Ex. 1. From a given point on a hyperbola, draw a straight line such that the segment intercepted between the other intersection with the hyperbola and a given asymptote, shall be equal to a given line.

When does the problem become impossible?

Ex. 2. The foot of the normal at P is equidistant from p, p'.

Ex. 3. Prove that $Qq \cdot Qq' = Pp^2$.

Ex. 4. If QK be drawn parallel to Cq' and $Q'K'$ parallel to Cq, then $Kq = K'Q'$, and $KQ = K'q'$.

Ex. 5. The tangent at P meets an asymptote in T, and a line TQ drawn parallel to the other asymptote meets the curve in Q; if PQ produced meets the asymptotes in R, R', prove that RR' is trisected at P and Q.

Ex. 6. The diameter bisecting any chord QQ' of a hyperbola meets the curve in P; and QH, PK, $Q'H'$ are drawn parallel to one asymptote meeting the other in H, K, H'. Prove that

$$CH \cdot CH' = CK^2.$$

Ex. 7. A line drawn through one of the vertices of a hyperbola, and terminated by two lines drawn through the other vertex parallel to the asymptotes, will be bisected at the other point where it cuts the hyperbola.

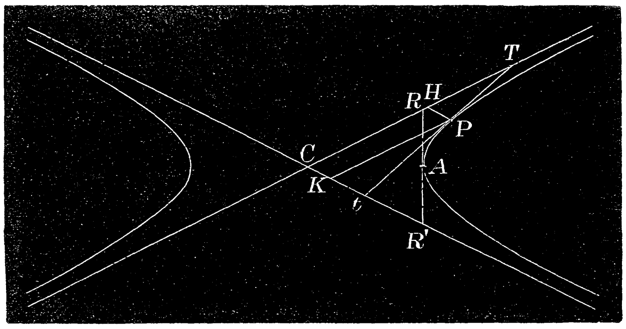
Ex. 8. If qT be the tangent from q, and QH, TK, $Q'H'$ be drawn parallel to Cq meeting Cq' in H, K, H', prove that

$$QH + Q'H' = 2TK.$$

Ex. 9. Through any point P on a hyperbola lines are drawn parallel to the asymptotes, meeting them in M and N; and any ellipse is constructed having CM, CN for semi-diameters. If CP cut the ellipse in Q, show that the tangent to the ellipse at Q is parallel to the tangent to the hyperbola at P. [Each is parallel to MN.]

* PROPOSITION XXXII.

The area of the triangle formed by the asymptotes and any tangent to a hyperbola is constant.



Let the tangents at the vertex A and at any point P meet the asymptotes in R, R' and T, t respectively.

Draw PH, PK parallel to the asymptotes, meeting them in H and K.

Then, since Tt is bisected at P,

$$CT = 2 \,.\, CH, \qquad \text{[Prop. XXXI.}$$

and $$Ct = 2 \,.\, CK, \qquad \text{[Euc. VI. 2.}$$

therefore $$CT \,.\, Ct = 4 \,.\, CK \,.\, CH$$
$$= 4 \,.\, PH \,.\, PK$$
$$= CS^2 \qquad \text{[Prop. XXIX.}$$
$$= CR \,.\, CR'. \qquad \text{[Def.}$$

Therefore the triangle CTt is equal to the triangle CRR', [Euc. VI. 15.

and is, therefore, constant.

Ex. 1. If any two tangents be drawn to a hyperbola, the lines joining the points where they met the asymptotes will be parallel.

Ex. 2. If TOt, $T'Ot'$ be two tangents meeting one asymptote in T, T', and the other in t, t', prove that

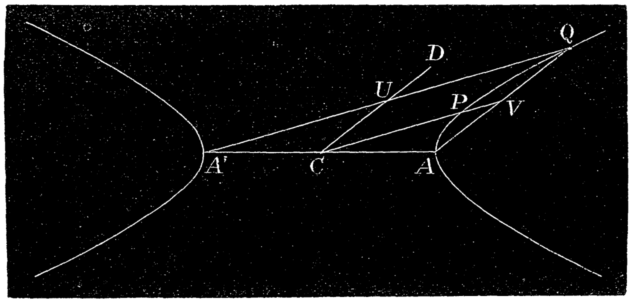
$$TO : Ot = t'O : T'O.$$

Ex. 3. Tangents are drawn to a hyperbola, and the portion of each tangent intercepted between the asymptotes is divided in a constant ratio. Prove that the locus of the points of section is a hyperbola. (Apply Prop. XXIX.)

PROPERTIES OF CONJUGATE DIAMETERS.

Proposition XXXIII.

If one diameter of a hyperbola bisects chords parallel to a second the second diameter bisects chords parallel to the first.



Let CP bisect chords parallel to CD, then CD bisects chords parallel to CP.

Draw AQ parallel to CD meeting CP produced in V. Join $A'Q$, intersecting CD in U.

Then, because AQ is bisected in V and AA' in C, CV is parallel to $A'Q$. [Euc. VI. 2.

Again, since AA' is bisected in C and CD is parallel to AQ, $A'Q$ is bisected by CD. [Euc. VI. 2.

Therefore CD bisects all chords parallel to $A'Q$,

[Prop. IX.

and, therefore, all chords parallel to CP.

Def. Two diameters so related that each bisects chords parallel to the other are called *conjugate diameters*.

Thus CP and CD are conjugate to each other; so also are the transverse and the conjugate axes.

It is clear that of two conjugate diameters, one (as CP) will meet the hyperbola, and the other (as CD) the conjugate hyperbola.

The *portion* CD terminated by the conjugate hyperbola is usually called the semi-diameter conjugate to CP.

Ex. 1. If any tangent to a hyperbola meet any two conjugate diameters, the rectangle under its segments is equal to the square of the parallel semi-diameter. (Cf. Chap. II., Prop. XXX., Ex. 7.)

Ex. 2. Given in magnitude and position any two conjugate semi-diameters of a hyperbola, find the transverse and conjugate axes. (Cf. Chap. II., Prop. XXX., Ex. 8.)

Ex. 3. Draw a tangent to a hyperbola parallel to a given straight line.

[The point of contact (P) of the required tangent is obtained by drawing CD parallel to the given straight line, and CP parallel to the tangent to the conjugate hyperbola at D.]

Ex. 4. If CQ be conjugate to the normal at P, CP is conjugate to the normal at Q.

Ex. 5. OP, OQ are tangents to a hyperbola from O. Prove that CO, PQ are parallel to a pair of conjugate diameters. (Prop. XIX.)

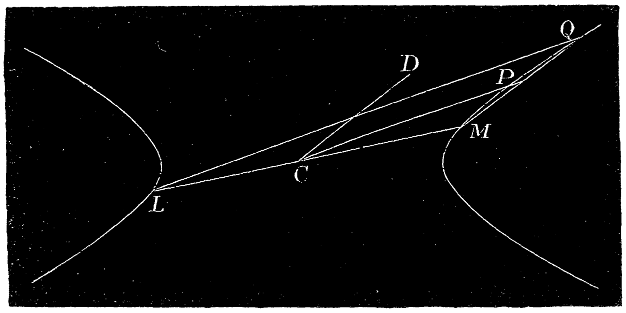
Ex. 6. An ellipse or a hyperbola is drawn touching the asymptotes of a given hyperbola. Prove that two of the chords of intersection of the curves are parallel to the chord of contact of the conic with the asymptotes.

[If PP' be the chord of contact and CV bisect PP', then CV, PP' are parallel to a pair of conjugate diameters in both conics.]

Def. Chords which join any point on a hyperbola to the extremities of a diameter are called *supplemental chords*.

PROPOSITION XXXIV.

Supplemental chords of a hyperbola are parallel to conjugate diameters.



Join any point Q on the hyperbola to the extremities of a diameter LCM. Then QL and QM are supplemental chords.

Draw CP, CD parallel to QL and QM respectively, then they shall be conjugate diameters.

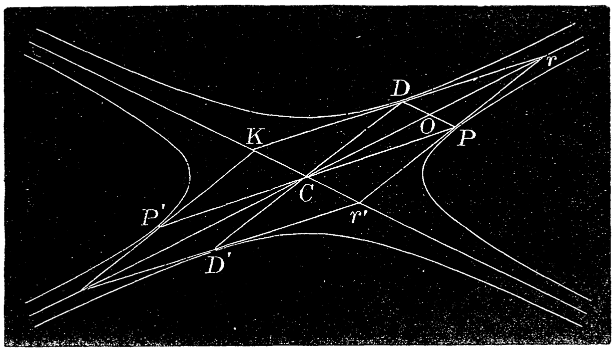
Because LM is bisected in C, and CP is parallel to LQ, CP produced bisects MQ, [Euc. VI. 2.
and, therefore, all chords parallel to CD. [Prop. IX.

Therefore CD bisects all chords parallel to CP, [Prop. XXXIII.
and is, therefore, conjugate to it.

PROPOSITION XXXV.

The tangents at the extremities of any pair of conjugate diameters meet on the asymptotes, and the line joining the extremities is parallel to one asymptote and bisected by the other.

Let CP, CD be a pair of conjugate semi-diameters. Draw rPr' the tangent at P, meeting the asymptotes in r and r'. Join Dr and produce rD to meet the other asymptote in K.



Now, since P is a point on the curve and D on its conjugate, and DC meets both the asymptotes in C and is parallel to Pr, [Props. XII. and XXXIII.

$$DC^2 = Pr \,.\, Pr' \qquad \text{[Prop. XXX.}$$

$$= Pr^2; \qquad \text{[Prop. XXXI.}$$

therefore $CD = Pr$.

Therefore Dr is parallel to CP, and Cr, PD bisect each other at O.

Again, since $Pr = Pr'$, [Prop. XXXI.

and $Or = OC$,

therefore PD is parallel to $r'K$. [Euc. VI. 2.

Therefore $Dr = DK$, [Euc. VI. 2.

and KDr is the tangent at D. [Prop. XXXI.

Ex. 1. If PD be drawn parallel to an asymptote to meet the conjugate hyperbola in D, CP, CD are conjugate diameters.

Ex. 2. Conjugate diameters of a hyperbola are also conjugate diameters of the conjugate hyperbola.

Ex. 3. CP, CD are conjugate diameters of a hyperbola. PN, DM are ordinates to the transverse axis. Prove that

$$\text{(i) } CM : PN = CA : CB.$$
$$\text{(ii) } DM : CN = CB : CA.$$

Let the tangent to the hyperbola at P and to the conjugate at D, meet the transverse axis in T, t respectively. Then CP, PT are parallel to Dt, DC. Now

$$CT \,.\, CN = CA^2 = Ct \,.\, CM. \quad \text{(Prop. XX.)}$$

$\therefore$ $$CM : CN = CT : Ct = PT : CD = PN : DM = CN : Mt\,;$$

$\therefore$ $$CN^2 = CM \,.\, Mt = CA^2 + CM^2. \quad \text{(Prop. XX.)}$$

$\therefore$ $$CM^2 = CN^2 - CA^2.$$

But $$PN^2 : CN^2 - CA^2 = CB^2 : CA^2. \quad \text{(Prop. VIII.)}$$

$\therefore$ (i) follows immediately.

Ex. 4. If the normal at P meet the axes in G, g, prove that

$$\text{(i) } PG : CD = CB : CA.$$
$$\text{(ii) } Pg : CD = CA : CB.$$
$$\text{(iii) } PG \,.\, Pg = CD^2.$$

[The triangles DCM and PGN are similar, as also the triangles DCM and Pgn.]

Ex. 5. A circle is drawn touching the transverse axis at C, and also touching the curve. Prove that the diameter conjugate to the diameter through either point of contact, is equal to SS'.

[If the normal at P meets the axes in G, g, and the tangent at P meets CB in t, $Ct = PG$, and $CD^2 = PG \,.\, Pg = Ct \,.\, Cg = CS^2$. Prop. XXIII., Ex. 1.]

Ex. 6. The area of the parallelogram formed by the tangents at the extremities of any pair of conjugate diameters, is constant and equal to $4 \,.\, CA \,.\, CB$. (Apply Prop. XXXII.)

Ex. 7. The tangent at a point P of an ellipse (centre O) meets the hyperbola having the same axes as the ellipse, in C and D. If Q be the middle point of CD, prove that OQ, OP are equally inclined to the axes.

[Draw OrR parallel to PQ, meeting the ellipse and hyperbola in r and R; then OP, Or are conjugate in the ellipse, and OQ, OR in the hyperbola. If PN, QM, rl, RL be the ordinates, we have, for the ellipse,

$$\frac{PN}{Ol} = \frac{OB}{OA} = \frac{rl}{ON}. \quad \text{(Chap. II., Prop. XXXIII.)}$$

or, $$\frac{PN}{ON} = \frac{OB^2}{OA^2} \cdot \frac{Ol}{rl}.$$

Similarly, for the hyperbola,

$$\frac{QM}{OM} = \frac{OB^2}{OA^2} \cdot \frac{Ol}{rl}. \quad \text{(Ex. 3.)}$$

$\therefore$ $$PN : ON = QM : OM.]$$

Ex. 8. With two conjugate diameters of an ellipse as asymptotes, a pair of conjugate hyperbolas is described. Prove that if the ellipse touch one hyperbola, it will also touch the other.

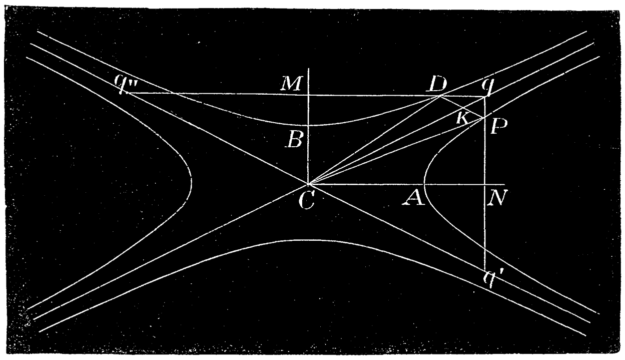
[The diameters drawn through the points of contact are conjugate to each other.]

Ex. 9. Apply this proposition to prove Prop. X.

PROPOSITION XXXVI.

The difference of the squares of any two conjugate semi-diameters of a hyperbola is constant

$$(CP^2 \sim CD^2 = CA^2 \sim CB^2).$$



Let CP, CD be a pair of conjugate semi-diameters.

Draw the ordinate $qPNq'$, meeting the asymptotes in q, q', and join PD; let PD meet the asymptote in K. Join Dq.

Then, since the asymptotes are equally inclined to the ordinate $qPNq'$, [Const.

and PK is parallel to the asymptote Cq', [Prop. XXXV.

the angles KqP and KPq are equal.

Therefore $Kq = KP = KD$. [Prop. XXXV.

Therefore the circle described on PD as diameter passes through q, and the angle PqD is a right angle. [Euc. III. 31.

If, therefore, qD produced meet the conjugate axis in M and the asymptote Cq' in q'', qMq'' will be at right angles to CB.

Now $Cq^2 - CP^2 = qN^2 - PN^2$ [Euc. I. 47.

$= Pq \,.\, Pq'$ [Euc. II. 5.

$= CB^2,$ [Prop. XXVIII.

and $Cq^2 - CD^2 = qM^2 - DM^2$ [Euc. I. 47.

$= Dq \,.\, Dq''$ [Euc. II. 5.

$= CA^2;$ [Prop. XXVIII.

therefore $CP^2 \sim CD^2 = CA^2 \sim CB^2.$

Ex. 1. If from any point on an asymptote of a hyperbola, ordinates be drawn to the curve and its conjugate, meeting them in P and D respectively, show that CP and CD will be conjugate semi-diameters, and conversely.

Ex. 2. Apply Prop. XXXV., Ex. 3, to prove this proposition.
We have $CN^2 - CM^2 = CA^2.$
Similarly, if Pn, Dm be ordinates to CB,
$Cm^2 - Cn^2 = CB^2,$
or $DM^2 - PN^2 = CB^2.$
Subtracting, $CP^2 \backsim CD^2 = CA^2 \backsim CB^2.$

Ex. 3. The difference between the sum of the squares of the distances of any point on the curve from the ends of any diameter, and the sum of the squares of its distances from the ends of the conjugate, is constant. [$= 2(CA^2 \backsim CB^2)$.]

Ex. 4. σ is the focus of the conjugate hyperbola lying on CB. Prove that $\sigma D - SP = CA - CB.$
(Apply Ex. 1, and Prop. XXVII., Ex. 5 and 13.)

Ex. 5. Prove that $SP \,.\, S'P = CD^2.$
[$SP \backsim S'P = 2 \,.\, CA$. Then square and substitute. Cf. also Prop. XXIII., Ex. 5, and Prop. XXXV., Ex. 3.]

Ex. 6. In Prop. XXIII., Ex. 1, prove that
$St : tg = CB : CD,$
CD being conjugate to CP. [Apply Ex. 5 and Prop. XXI.]

Ex. 7. If the tangent at P meet any conjugate diameters in T and t, the triangles SPT, $S'Pt$ are similar.
[$SP : PT = Pt : S'P$. Apply Ex. 5 and Prop. XXXIII., Ex. 1.]

Ex. 8. If the tangent at P meet the conjugate axis in t, the areas of the triangles SPS', StS' are the ratio of $CD^2 : St^2$. (Apply Prop. XXIII., Ex. 1.)

Ex. 9. Through C a line is drawn parallel to either focal distance of P; if DE is drawn perpendicular to this line, prove that $DE = CB$.

[If SY is perpendicular to the tangent at P, the triangles SYP, CDE are similar. Then

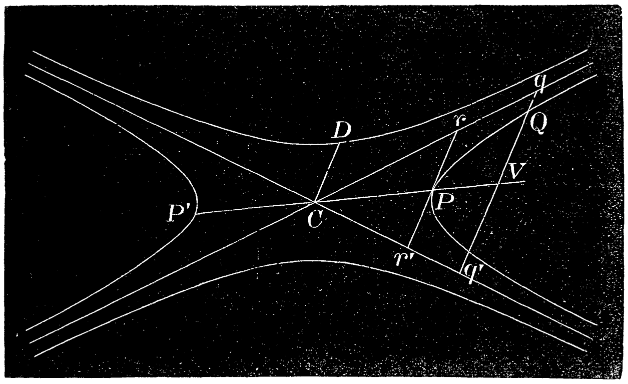
$$DE : CD = SY : SP = S'Y' : S'P;$$

$$\therefore \quad \frac{DE^2}{CD^2} = \frac{SY \cdot S'Y'}{SP \cdot S'P} = \frac{BC^2}{CD^2}. \quad \text{Prop. XXI. and Ex. 5.]}$$

* PROPOSITION XXXVII.

The square of the ordinate of any point of a hyperbola with respect to any diameter varies as the rectangle under the segments of the diameter made by the ordinate.

$$(QV^2 : PV \cdot P'V = CD^2 : CP^2.)$$



Let QV be an ordinate to the diameter PCP', meeting the asymptotes in q, q'.

Draw the tangent at P meeting the asymptotes in r, r'.

Then Pr is parallel to QV. [Prop. XII.

Therefore, by similar triangles,

$$qV^2 : Pr^2 = CV^2 : CP^2,$$

therefore $$qV^2 - Pr^2 : Pr^2 = CV^2 - CP^2 : CP^2,$$

but $$Pr\,.\,Pr' = Qq\,.\,Qq',$$ [Prop. XXX.

or $$Pr^2 = qV^2 - QV^2,$$ [Prop. XXXI. and Euc. II. 5.

therefore $$qV^2 - Pr^2 = QV^2.$$

Also $$CV^2 - CP^2 = PV\,.\,P'V,$$ [Euc. II. 5.

therefore $$QV^2 : Pr^2 = PV\,.\,P'V : CP^2,$$

or $$QV^2 : PV\,.\,P'V = Pr^2 : CP^2,$$

which is constant.

Since $$CD^2 = Pr\,.\,Pr'$$ [Prop. XXX.

$$= Pr^2,$$ [Prop. XXXI.

this result may also be expressed as

$$QV^2 : PV\,.\,P'V = CD^2 : CP^2.$$

Ex. If the tangent at D to the conjugate hyperbola meet an asymptote in r' and the hyperbola in q', and the ordinate vq' parallel to the tangent at P be produced to meet the same asymptote in R, show that $\triangle\, CPr' = \frac{1}{2} \triangle\, CvR$.

THE EQUILATERAL HYPERBOLA.

The rectangle contained by the transverse axis of a central conic and its latus rectum has been called by Apollonius the "figure of the conic upon its axis." It is evident that the "minor" or "conjugate" axis of a central conic, according as it is an ellipse or a hyperbola, is equal to the side of a square equivalent in area to the "figure." (Chap. II., Prop. VI., and Chap. III., Prop. V.)

A hyperbola which has the sides of its "figure" equal is called an *equilateral* hyperbola. The latus rectum being thus equal to the transverse axis it is clear that the conjugate axis is equal to the transverse axis (Chap. III., Prop. V.); in other words the two axes of an *equilateral* hyperbola are equal.

From Prop. XXVII. it is clear that the asymptotes of an *equilateral* hyperbola are at right angles to each other. From this property the curve is also called a *rectangular* hyperbola.

Ex. Prove that the locus of the intersection of tangents to a parabola including half a right angle, is a rectangular hyperbola. (Prop. I., Ex. 10, and Prop. XXVII., Ex. 3.)

The properties of the hyperbola proved in the preceding propositions are, of course, true for the equilateral hyperbola as well. In some cases, however, the results assume forms which are deserving of notice.

Thus, for the equilateral hyperbola, we have

Prop. III. $e=\sqrt{2}$, (See Ex. 2.)

$$CS^2=2CA^2,$$
$$CS=2CX.$$

Ex. If a circle be described on SS' as diameter, the tangents at the vertices will intersect the asymptotes in the circumference.

Prop. V. $SL=CA$,

or, Latus rectum $=AA'$.

Prop. VIII. $PN^2=AN.A'N.$

Ex. 1. If PNP' be a double ordinate, the angles PAP' and $PA'P'$ are supplementary.

Ex. 2. The triangle formed by the tangent at any point and its intercepts on the axes, is similar to the triangle formed by the central radius to that point and the abscissa and ordinate of the point. (See Prop. XX., Ex. 1.)

Ex. 3. If M be a point on the conjugate axis, and MP be drawn parallel to the transverse axis meeting the curve in P, then $PM=AM$.

Ex. 4. The tangent at any point P of a circle meets a fixed diameter AB produced in T, show that the straight line through T perpendicular to AB meets AP BP produced in points which lie on an equilateral hyperbola.

Ex. 5. If AB be any diameter of a circle and PNQ an ordinate to it, the locus of intersection of AP, BQ is an equilateral hyperbola.

Ex. 6. The locus of the point of intersection of tangents to an ellipse which make equal angles with the major and minor axis respectively, and are not at right angles, is a rectangular hyperbola. (The foci of the ellipse will be the vertices.)

Prop. XXVI. $CN = NG$,
$PG = Pg = CP$.

Prop. XXXI. $CP = Pr = Pr'$.

Ex. 1. A circle whose centre is any point P and radius CP, intersects the normal on the axes and the tangent on the asymptotes.

Ex. 2. If the tangents at two points Q and Q' meet in T, and if CQ, CQ' meet these tangents in R and R', the circle circumscribing RTR' passes through C.

Ex. 3. The angle subtended by any chord at the centre is the supplement of the angle between the tangents at the ends of the chord.

PROPOSITION A.

Conjugate diameters are equal in the equilateral hyperbola and the asymptotes bisect the angle between them.

Let CP, CD be any two conjugate semi-diameters. Then $CP^2 \sim CD^2 = CA^2 \sim CB^2 = O$, [Prop. XXXVI. since the axes are equal.

Therefore $CP = CD$.

Again, since the asymptote Cr (Fig., Prop. XXXV.) bisects PD it must bisect the angle PCD.

Similarly, it may be shown that the asymptote Cr' bisects the angle PCD'.

Ex. 1. A circle is described on the transverse axis as diameter. Prove that if any tangent be drawn to the hyperbola, the straight lines joining the centre of the hyperbola with the point of contact and with the middle point of the chord of intersection of the tangent with the circle, are inclined to the asymptotes at complementary angles.

Ex. 2. The lines drawn from any point on the curve to the extremities of any diameter make equal angles with the asymptotes. (Prop. XXXIV.)

Ex. 3. The focal chords drawn parallel to conjugate diameters are equal. (Props. VI. and X.)

Ex. 4. If two concentric rectangular hyperbolas be described, the axes of one being the asymptotes of the other, they will cut at right angles.

Ex. 5. The normals at the ends of two conjugate diameters intersect on the asymptote and are parallel to another pair of conjugate diameters. (Prop. XXXV.)

Ex. 6. If QV be an ordinate of a diameter PCp,

$$QV^2 = PV.pV. \qquad \text{(Prop. XXXVII.}$$

Ex. 7. If tangents parallel to a given direction are drawn to a system of circles passing through two fixed points, the points of contact lie on a rectangular hyperbola. (Apply Ex. 6.)

Ex. 8. Given the base of a triangle and the difference of the angles at the base, prove that the locus of the vertex is a rectangular hyperbola. (Apply Ex. 6.)

Ex. 9. PCp is a diameter and QV an ordinate, prove that QV is the tangent at Q to the circle round the triangle PQp. (Apply Ex. 6.)

Ex. 10. If P be a point on an equilateral hyperbola and if the tangent at Q meet CP in T, the circle circumscribing CTQ touches the ordinate QV conjugate to CP. (Apply Ex. 6 and Prop. XX.)

Ex. 11. The angle between a chord PQ and the tangent at P, is equal to the angle subtended by PQ at the other extremity of the diameter through P.

Ex. 12. The distance of any point on the curve from the centre is a geometric mean between its distances from the foci. (Apply Prop. XXXVI., Ex. 5.)

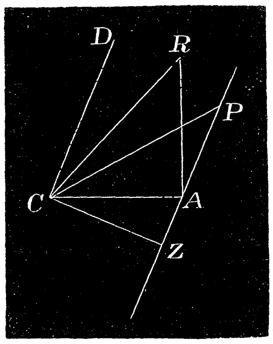
Ex. 13. The points of intersection of an ellipse and a confocal rectangular hyperbola are the extremities of the equi-conjugate diameters of the ellipse. (Apply Prop. XXXVI., Ex. 5, and Chap. II., Prop. XXXV., Ex. 5.)

Ex. 14. If two focal chords be parallel to conjugate diameters, the lines joining their extremities intersect on the asymptotes.

[If PSp, QSq be the chords, it may be shown that pq, PQ and an asymptote will meet on the directrix at the same point. Prop. VII. and Prop. XXVII., Ex. 5.]

PROPOSITION B.

In the equilateral hyperbola the transverse axis bisects the angle between the central radius vector of any point and the central perpendicular on the tangent at that point.



Let P be any point on an equilateral hyperbola and CD the semi-diameter conjugate to CP; let CZ be the perpendicular on the tangent at P.

If CR be the asymptote, because

$$CA = AR, \qquad \text{[Prop. XXVII.}$$

the angle ACR is half a right angle, that is, half of the angle DCZ, since CD is parallel to PZ.

[Props. XII. and XXXIII.

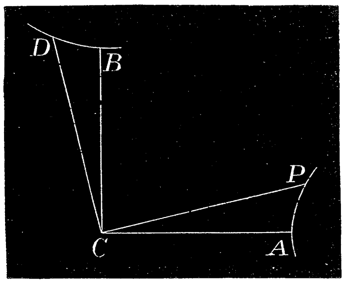
But the angle PCR is half of the angle PCD; [Prop. A. therefore the remaining angle PCA is half of the remaining angle PCZ, that is, CA bisects the angle PCZ.

Ex. 1. Prove that $CZ . CP = CA^2$. (Apply Prop. XX.)

Ex. 2. Prove that the angles CPA and CAZ are equal.

PROPOSITION C.

In the equilateral hyperbola diameters at right angles to each other are equal.



Let there be two semi-diameters CP, CD at right angles to each other, meeting the curve and its conjugate in P and D respectively.

Then the angle $ACB =$ the angle PCD,
each being a right angle. Taking away the common angle PCB,

the angle $ACP =$ the angle BCD.

Hence from symmetry, since the curve and its conjugate are equal and similarly placed with respect to the axes,

$$CP = CD.$$

Ex. 1. Prove that focal chords at right angles to each other are equal.

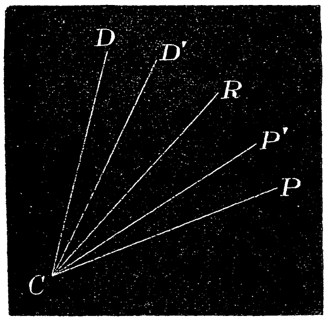
Ex. 2. If a right-angled triangle be inscribed in the curve, the normal at the right angle is parallel to the hypotenuse. (See Prop. X.)

Ex. 3. Chords which subtend a right angle at a point P of the curve, are all parallel to the normal at P.

PROPOSITION D.

The angle between any two diameters of an equilateral hyperbola is equal to the angle between their conjugates.

Let CP, CP' be any two semi-diameters, and CD, CD' the semi-diameters conjugate to them respectively.



Then, if CR be the asymptote,

the angle PCR = the angle DCR, [Prop. A.

and the angle $P'CR$ = the angle $D'CR$; [Prop. A.

therefore, by subtraction,

the angle PCP' = the angle DCD'.

Ex. 1. Conjugate diameters are inclined to either axes at angles which are complementary.

Ex. 2. If CP, CD be conjugate semi-diameters and PN, DM ordinates, the triangles PCN, DCM are equal in all respects.

Ex. 3. The difference between the angles which the lines joining any point on the curve to the extremities of a diameter make with the diameter, is equal to the angle which the diameter makes with its conjugate.

Ex. 4. The angles subtended by any chord at the extremities of a diameter are equal or supplementary. (Apply Prop. XXXIV.)

Ex. 5. AB is a chord of a circle and a diameter of a rectangular hyperbola, P is any point on the circle, AP, BP, produced if necessary, meet the hyperbola in Q, Q' respectively. Prove that BQ and AQ' intersect on the circle. (Apply Ex. 4.)

Ex. 6. A circle and a rectangular hyperbola intersect in four points and one of their common chords is a diameter of the hyperbola. Show that the other common chord is a diameter of the circle. (Apply Ex. 4.)

Ex. 7. QN is drawn perpendicular from any point Q on the curve to the tangent at P. Prove that the circle round CNP bisects PQ. (Apply Ex. 4.)

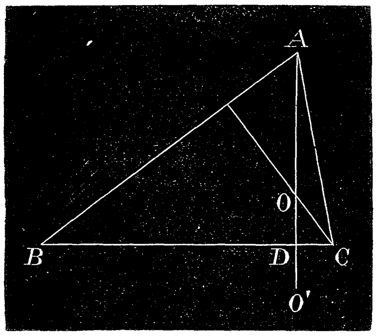
Ex. 8. If a rectangular hyperbola circumscribe a triangle, the locus of its centre is the nine-point circle.

[The diameters to the middle points of the sides are conjugate to the sides respectively.]

Ex. 9. The tangent at a point P of a rectangular hyperbola meets a diameter QCQ' in T. Prove that CQ and TQ' subtend equal angles at P.

* PROPOSITION E.

If a rectangular hyperbola circumscribe a triangle it passes through the orthocentre.



Let a rectangular hyperbola circumscribing a triangle ABC meet AD, drawn perpendicular to BC, in O.

Then the rectangles $AD \cdot OD$, $BD \cdot CD$ are as the squares of the semi-diameters parallel to AD, BC. [Prop. X. But the semi-diameters being at right angles to each other, are equal; [Prop. C.

therefore $AD \cdot OD = BD \cdot CD$.

Therefore, as is well known, the point O must coincide either with the orthocentre or with the point O' where AD meets the circle circumscribing the triangle ABC.

But the latter case is impossible; for then the lines *AD, BC,* which are at right angles to each other, will be equally inclined to the axis, [Prop. XI. and will, therefore, be parallel to the asymptotes, which are also at right angles to each other and equally inclined to the axis. [Prop. XXVII. Hence *BC,* being parallel to an asymptote, cannot meet the curve in two points (see Prop. XXVII., Ex. 2), which is contrary to the hypothesis.

Hence the curve must pass through the orthocentre.

Ex. 1. Every conic passing through the centres of the four circles which touch the sides of a triangle is a rectangular hyperbola.

Ex. 2. Any conic passing through the four points of intersection of two rectangular hyperbolas, is itself a rectangular hyperbola.

Ex. 3. If two rectangular hyperbolas intersect in A, B, C, D, the circles described on AB, CD as diameters intersect each other orthogonally.

[D is the orthocentre of the triangle ABC. Observe that the distance between the middle points of AB and CD is equal to the radius of the circumscribing circle.]

MISCELLANEOUS EXAMPLES ON THE HYPERBOLA.

1. Given the two asymptotes and a point on the curve, show how to construct the curve and find the position of the foci.

2. CP, CD are conjugate semi-diameters and the tangent at P meets an asymptote in r. If rn be the perpendicular from r on the transverse axis DPn is a right line.

3. P is any point on a hyperbola whose foci are S, S'; if the tangent at P meet an asymptote in T the angle between that asymptote and $S'P$ is double the angle STP.

ellipses. Prove that the locus of the points of contact is a rectangular hyperbola.

[See figure, Chap. II., Prop. XXVIII. $CF \propto CG$ and $PF \propto PR \propto Ct \propto CT$. Therefore $PF.CF \propto CG.CT = CS^2 =$ constant.]

35. From the point of intersection of the directrix with one of the asymptotes of a rectangular hyperbola a tangent is drawn to the curve, meeting the other asymptote in T. Prove that CT is equal to the transverse axis. (Apply Prop. XXXII. and Prop. XXVII., Ex. 5.)

36. If a rectangular hyperbola, having its asymptotes coincident with the axes of an ellipse, touch the ellipse, the axis of the hyperbola is a mean proportional between the axes of the ellipse. (Apply Props. XXXI., XXXII., and XX.)

37. Ellipses are inscribed in a given parallelogram; prove that their foci lie on a rectangular hyperbola.

38. Given the centre, a tangent, and a point on a rectangular hyperbola, find the asymptotes.

39. Prove that the parallel focal chords of conjugate hyperbolas are to one another as the eccentricities of the hyperbolas.

40. With each pair of three given points as foci a hyperbola is drawn passing through the third point. Prove that the three hyperbolas thus drawn intersect in a point.

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